The Quantum State of a Propagating Laser Field

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Optical implementations of quantum communication protocols typically involve laser fields. However, the standard description of the quantum state of a laser field is surprisingly insufficient to understand the quantum nature of such implementations. In this paper, we give a quantum information-theoretic description of a propagating continuous-wave laser field and reinterpret various quantum-optical experiments in light of this. A timely example is found in a recent controversy about the quantum teleportation of continuous variables. We show that contrary to the claims of T. Rudolph and B. C. Sanders [Phys. Rev. Lett. **87**, 077903 (2001)], a conventional laser can be used for quantum teleportation with continuous variables and for generating continuous-variable quantum entanglement. Furthermore, we show that optical coherent states do play a privileged role in the description of propagating laser fields even though they cannot be ascribed such a role for the intracavity field.

I. INTRODUCTION

What is the quantum state of a laser field? According to textbook laser theory—see for example Chapter 17 in Ref. [1] or Chapter 12 in Ref. [2]—the quantum state of the field *inside* the laser cavity in a steady state is a mixed state diagonal in the photon-number basis, with Poissonian number statistics:

$$\rho_{|\alpha|} = e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!} |n\rangle\langle n| \, . \tag{1}$$

Such a state does not have a well-defined phase. Yet, laser fields are routinely used to define a phase standard. Moreover, many, if not all, standard optics experiments seem to be consistent with the assumption that the laser field is in a pure coherent state.

The common explanation for how this comes about seems most often to rest on the mathematical identity

$$e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!} |n\rangle\langle n| = \int \frac{\mathrm{d}\varphi}{2\pi} |\alpha e^{i\varphi}\rangle\langle \alpha e^{i\varphi}| \, , \tag{2}$$

where the right-hand side denotes an integral over all coherent states $|\alpha e^{i\varphi}\rangle$ with amplitude $|\alpha|$. Through this identity one might think at first sight that one has the right to think of laser light as a mixture of coherent states: "The ideal laser field really is in some coherent state $|\alpha e^{i\varphi}\rangle$, we just do not know which one." (See for instance, Ref. [2, pp. 15, 38].) However, this kind of thinking already carries the seed of its own demise. For if one can use this argument to imagine the existence of an unknown coherent state for describing the true state of the laser field, then one can use it just as well to imagine the existence of an unknown number state for the same task. To think otherwise is to commit the so-called *preferred ensemble fallacy* (PEF), a move that has no justification within standard quantum mechanics. (For a general discussion of the PEF, see Ref. [3]. For some other examples of the havoc it can cause in quantum information science, see Refs. [4,5].)

What is the solution to this conundrum? Of course, one can argue that any experiment whose outcome does not depend on the absolute phase φ cannot distinguish between a pure-state $|\alpha e^{i\varphi}\rangle$ and a mixed-state $\rho_{|\alpha|}$. However, this observation is not general enough to explain many of the results of present-day optics experiments. For instance, it does not yet explain why a phase measurement between two independent laser beams will give the same result as a subsequent phase measurement on more light emanating from the same lasers, or how coherent manipulations of atoms and ions are in fact possible even though no coherent superpositions of atomic states are produced. It also may not be clear what is meant by phase diffusion in the context of laser theory—see for example Chapter 20 of Ref. [1]—if the phase of the laser cavity field is totally random.

Mølmer was perhaps the first to address the apparent contradiction between the two different descriptions of a laser field in Ref. [6]. In particular, he studied a standard measurement of the phase between two independent light beams emanating from cavities initially in pure *number states*, so that neither state has a well-defined phase by itself nor relative to the other. He showed that indeed a definite phase difference will be measured; it is just that that value will be random from experiment to experiment. Nevertheless, if one keeps monitoring the phase difference between two such light beams, the randomly established phase value will persist.

In this paper, we contribute to the furtherance of this discussion by diverting attention away from the intracavity field and refocusing it on the field after it leaks out of the cavity, i.e., on the *propagating* laser beam. In particular, we reexamine the question of the quantum state of a propagating laser field from a quantum-information theoretic perspective. Using this description, the solution to the conundrum above becomes clear. Moreover, it also allows us to answer an important question raised in Ref. [7]. There it is concluded that teleportation with continuous variables is not possible with a mixed state of the form (1), but requires a true coherent state. The main reason for their conclusion is the contention that a mixture of two-mode squeezed states produced by a laser in a mixed state does not contain any quantum entanglement. This is an important observation. In fact, this is a splendid example of why Eq. (2) does not capture the complete essence of most experiments with laser light. Our formulation clarifies why the coherent state plays a privileged and unique role in the description of propagating laser fields, and how a conventional laser can produce quantum entanglement, even if it cannot actually produce a two-mode squeezed state.

The plan of the remainder of the paper is as follows. In Section II, we use the input-output formalism of Refs. [2,8] to derive a multi-mode description of an ideal laser beam. We then use the quantum de Finetti theorem [9,10] to show in what sense the expression just derived is a unique one. In Section III, we apply the results of Section II to elucidate several examples: measurement of the relative phase between two independent lasers, the coherent excitation of a two-level atom, the production and detection of squeezed states, the production of two-mode squeezed states, and finally quantum teleportation of continuous variables. In Section IV, we give some concluding remarks.

For all of the examples addressed in Section III the description of an *ideal* noiseless laser is sufficient for understanding the point of principle. It is nevertheless interesting and important to describe phase diffusion and its effects on the quantum state of a laser beam. To that end, we further give a discussion of the phase-diffusing laser in light of the present considerations in Section IIC. Remarks are also made throughout Section III concerning the effect of phase diffusion for the phenomenon at hand.

II. THE QUANTUM STATE OF A PROPAGATING LASER BEAM

We are interested in calculating the quantum state of the light field of a continuous-wave (CW) laser. We model the laser as a one-sided cavity driven by a constant force (a voltage or an external field) far above threshold. We first consider an imaginary case where the field inside the cavity is in a coherent state and calculate the quantum state of the field outside the laser cavity. Then, using the identity (2), we use the linearity of quantum mechanics to derive the true state of a laser field outside of the cavity. Subsequently we imagine that we partition the light beam into packages of equal length (or duration) and rewrite the result in terms of the quantum states of the individual packages. We then compare that result with the general form of the quantum state of an ensemble produced by a source that emits unknown but identical states. Next we consider a more realistic model of a laser and include phase diffusion.

A. Input-Output Relations

We employ standard input-output theory [2,8] to connect the quantum field inside a laser cavity to its output field. First, we separate the field modes into two parts. A single-mode annihilation operator a describes the field with frequency ω_0 inside the cavity; continuous-mode operators $b(\omega)$ describe modes with frequency ω outside the cavity. This separation is an approximation. It is valid for (1) high-finesse cavities (i.e., the cavity decay rate κ must be much smaller than the distance between adjacent resonant frequencies, $\pi c/L$, with L the length of the cavity) (2) on time scales much longer than a cavity round trip time L/c and (3) for frequencies near resonance (i.e., for frequencies ω within a few κ from the relevant resonance frequency, ω_0, in the problem) [11].

We define input and output operators by

$$\begin{aligned} a_{\rm in}(t) &= \frac{-1}{\sqrt{2\pi}} \int {\rm d}\omega e^{-i\omega(t-t_0)} b_0(\omega), \\ a_{\rm out}(t) &= \frac{1}{\sqrt{2\pi}} \int {\rm d}\omega e^{-i\omega(t-t_1)} b_1(\omega), \end{aligned} \tag{3}$$

where $t_0 \to -\infty$ is a time in the far past and $t_1 \to \infty$ is a time in the far future. The operators $b_0(\omega)$ and $b_1(\omega)$ are defined to be the Heisenberg operators $b(\omega)$ at times $t = t_0$ and $t = t_1$, respectively. The input and output operators $a_{\rm in,out}(t)$ are not themselves Heisenberg operators. They do satisfy, by construction, bosonic commutation relations for free-field continuous-mode operators,

$$[a_{\rm in,out}(t), a^\dagger_{\rm in,out}(t')] = \delta(t-t'). \tag{4}$$

The integrations over frequencies ω extend from $-\infty$ to $+\infty$. This, obviously, involves an approximation. Namely ω is in fact defined as the frequency relative to the relevant resonance frequency we are interested in (i.e., we moved to a rotating frame), and thus ranges in principle from $-\omega_0$ to ∞. Since ω_0 is usually by far the largest frequency in the problem, it is a good approximation to extend the limit down to $-\infty$. This is valid as long as we consider times scales much larger than $1/\omega_0$.

Since nonlinear effects are typically very small, the interaction between the cavity mode and the outside modes can be well approximated by a linear interaction. Using the fact that the coupling (the cavity decay rate) is more or less constant over the relevant range of frequencies, one may choose the coupling to be constant over the entire range of frequencies to good approximation. The resulting input-output relation

$$a_{\rm in}(t) + a_{\rm out}(t) = \sqrt{\kappa} a(t), \tag{5}$$

with κ the decay rate of the cavity takes then a simple form, which can in fact be interpreted as a boundary condition on the electric field. It is important to note that this equation is valid irrespective of the internal dynamics of the cavity mode.

B. The Noiseless Laser

In this subsection we neglect dissipation and noise due to, e.g., spontaneous emission in the atomic laser medium. In this case the field inside the laser cavity is well approximated when the laser operates far above threshold by a state of the form (2).

When the input field is the vacuum and the field inside the cavity is for the moment assumed to be a coherent state $|\alpha e^{i\phi}\rangle$ with a known phase, then according to (5) the output field is an eigenstate of $a_{\rm out}$ with eigenvalue $\beta(t) \equiv \sqrt{\kappa}\alpha e^{i\phi}$. Such a state is a continuous-mode coherent state [12] and can be written in the Schrödinger picture as

$$|\{\beta(t)\}\rangle \equiv \exp\left(\int {\rm d}\omega[\beta(\omega)b^\dagger(\omega) - \beta^*(\omega)b(\omega)]\right)|{\rm vac}\rangle, \tag{6}$$

with $|{\rm vac}\rangle$ is the vacuum state and $\beta(\omega)$ is the Fourier transform of $\beta(t)$. In the idealized case without noise one simply has a stationary laser beam with fixed frequency ω_0. A continuous-mode coherent state can be described alternatively as an infinite tensor product of discrete-mode coherent states [12]. Define a complete set of functions $\{\Phi_i(t)\}$ satisfying the following orthogonality and completeness relations,

$$\int {\rm d}\tau \Phi_i(\tau)\Phi_j^*(\tau) = \delta_{ij}, \tag{7}$$

and

$$\sum_i \Phi_i(t)\Phi_i^*(t') = \delta(t-t'). \tag{8}$$

We may then define annihilation operators c_i (satisfying the correct bosonic commutation relations for discrete operators) according to

$$c_i = \int {\rm d}t \Phi_i^*(t) a_{\rm out}(t). \tag{9}$$

An eigenstate of $a_{\rm out}(t)$ with eigenvalue $\beta(t)$ is also an eigenstate of c_i with eigenvalue

$$\alpha_i = \int {\rm d}t \Phi_i^*(t)\beta(t). \tag{10}$$

We now apply this formalism to describe laser light as a sequence of packets of light, each with the same duration T. We thus define a set of functions $\{\Psi_n(t)\}$ by

$$\Psi_n(t) = \begin{cases} \dfrac{1}{\sqrt{T}} & \text{for } \left|t - \dfrac{z_0}{c} - nT\right| < \dfrac{T}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

The label z_0 refers to an arbitrarily chosen reference position z_0 relative to which we partitioned the light beam into equal pieces of length cT (see Figure 1). This set of functions is orthogonal and can be extended to form a complete set satisfying (7) and (8). For a CW laser described by $\beta(t) = \sqrt{\kappa}\alpha e^{i\phi}$ we see that each part n of the light beam is in the same coherent state with eigenvalue

$$\alpha_n = \sqrt{\kappa T}\alpha e^{i\phi} \equiv \alpha_0, \tag{12}$$

corresponding to the modes described by (11), and $\alpha_i = 0$ for all other modes. The duration T must be much larger than both $1/\omega_0$ and L/c but can be arbitrarily chosen otherwise.

Now assuming that the field inside the laser cavity is in fact a mixture $\rho_{|\alpha|}$, the quantum state of a sequence of N parts corresponding to the set $\{\Psi_n\}$ is thus

$$\tilde{\rho}_N = \int \frac{{\rm d}\varphi}{2\pi}\left(|\alpha_0 e^{i\varphi}\rangle\langle\alpha_0 e^{i\varphi}|\right)^{\otimes N}, \tag{13}$$

where the integrand signifies an N-fold tensor product over the separate packets.

C. The Phase-Diffusing Laser

Equation (13) is the quantum state of an ideal propagating laser field. Now let us consider a more realistic laser and take into account the effects of noise and dissipation. We present here only the main ideas and results from standard laser theory and use those to connect the field inside the laser cavity to the outside field we are interested in. For technical details we refer the reader to Ref. [1].

A first consequence of the presence of noise is that the state of the laser cavity field is not a steady state. Instead, there is in general a decay towards a steady state. The diagonal matrix elements (in the number-state basis) of the density matrix of the field decay towards a Poissonian distribution, the off-diagonal matrix elements decay to zero. The steady state at any time is, therefore, still a mixture of coherent states with random phases. The rate at which that steady state is reached is proportional to the average number of photons inside the cavity and inversely proportional to the quality factor of the cavity. Far above threshold and for high-quality cavities we may then approximate the state of the cavity field by its steady state value, as long as we consider time scales longer than the time needed to reach equilibrium.

For the purpose of finding the quantum state of the field outside the laser the most convenient representation is the Heisenberg picture, so that we can apply the input-output relation (5). Chapter 20 of Ref. [1] derives an equation for the operator $a(t)$. When the atomic operators have been eliminated the resulting equation for $a(t)$ turns out to be nonlinear (it contains a term proportional to $\langle a^\dagger(t)a(t)\rangle a(t)$) and to contain fluctuating noise terms. This equation is hard to solve in general but far above threshold one may go to the “classical” limit and write down an equation for the expectation value of $a(t)$ while using a decorrelation approximation that replaces $\langle a^\dagger(t)a(t)\rangle$ with $|\langle a(t)\rangle|^2$. This is basically the same approximation as the steady-state approximation mentioned above, except that it implicitly assumes, for the moment, that the steady state is a coherent state, rather than a mixture of coherent states. The intermediate result then gives us a coherent state whose amplitude and phase fluctuate randomly with time. However, the phase fluctuations are typically much larger than the amplitude fluctuations and this leads to the concept of a phase-diffusing laser. The amplitude of the field in the cavity is approximated to be $\alpha e^{i\phi}e^{i\eta(t)}$ with $\eta(t)$ a random Gaussian variable with correlations given by

$$\langle a^\dagger(t)a(0)\rangle = \alpha^2 \exp(-Dt), \tag{14}$$

with D the phase diffusion constant (determined by spontaneous emission rate of the atomic medium, the quality factor of the laser cavity and the average number of photons). According to (5) the outside field is still a continuous-mode coherent state with amplitude $\beta(t) = \sqrt{\kappa}\alpha e^{i\phi}e^{i\eta(t)}$, but it is no longer monochromatic since β is not constant. Indeed, the laser linewidth is given by D. Nevertheless, we can still use the same complete set of functions $\Phi_i(t)$ of which the functions $\Psi_n(t)$ of Eq. (11) are a subset. There are two types of modifications to the result (13). First, the modes corresponding to $\Psi_n(t)$ are no longer the only ones with a nonzero amplitude, and second, the amplitudes α_n,

$$\alpha_n = \int \mathrm{d}t \Psi_n^*(t)\beta(t), \tag{15}$$

are no longer constant in magnitude, nor do they all have the same phase. The absolute value of the amplitude is in fact reduced since

$$|\alpha_n| = \left| \int_{z_0/c+(n-1/2)T}^{z_0/c+(n+1/2)T} \mathrm{d}t e^{i\eta(t)} \right| \frac{\alpha}{\sqrt{T}}, \tag{16}$$

and the integral over time is less than T. This is consistent with the fact that other modes carry a finite amount of light as well.

If we choose T to be much smaller than a diffusion time $1/D$ (but still larger than an optical period and a cavity roundtrip time), then the amplitudes $|\alpha_n|$ are almost equal to what they would be in the absence of noise since the integral appearing in (16) is almost equal to T. Hence, in that case the quantum state of the laser is well approximated by

$$\begin{aligned}\tilde{\rho}_N = \int \mathrm{d}\epsilon_1 \int \mathrm{d}\epsilon_2 \cdots \int \mathrm{d}\epsilon_k \int \frac{\mathrm{d}\varphi}{2\pi} |\alpha_0 e^{i\varphi}\rangle\langle\alpha_0 e^{i\varphi}| \\ \otimes P(\epsilon_1)|\alpha_0 e^{i\varphi+i\epsilon_1}\rangle\langle\alpha_0 e^{i\varphi+i\epsilon_1}| \\ \otimes P(\epsilon_2)|\alpha_0 e^{i\varphi+i\epsilon_1+i\epsilon_2}\rangle\langle\alpha_0 e^{i\varphi+i\epsilon_1+i\epsilon_2}| \ \cdots \\ \otimes P(\epsilon_k)|\alpha_0 e^{i\varphi+i\sum_{k=1}^N \epsilon_k}\rangle\langle\alpha_0 e^{i\varphi+i\sum_{k=1}^N \epsilon_k}|,\end{aligned} \tag{17}$$

for the modes corresponding to Ψ_n with the remaining modes almost empty. Here we introduced the random variables ϵ_k by

$$\epsilon_k = \mathrm{Im}\left[\log \int_{z_0/c+(k-1/2)T}^{z_0/c+(k+1/2)T} \mathrm{d}t e^{i\eta(t)}\right], \tag{18}$$

with average zero and variance $\sigma = \sqrt{2DT} \ll 1$, and the corresponding probability distribution

$$P(\epsilon) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\epsilon^2}{2\sigma^2}\right). \tag{19}$$

D. The Quantum de Finetti Theorem

The result (13) for an ideal propagating laser field displays an apparent privileged role for coherent states in describing a propagating laser field: Although the quantum state inside the laser is a mixed state diagonal in the number-state basis, the quantum state of the output is not equal to a product of mixed states $(\rho_{|\alpha_0|})^{\otimes N}$ (as it would be for a pulsed laser). Rather it is a mixture of N copies of a coherent state, each copy with the same "unknown" phase. The real question is, is this the only such description? The last thing we would want to do is commit the *preferred ensemble fallacy* (PEF) that Rudolph and Sanders [7] rightly warn against. In other words, in analogy to Eq. (2), how do we know that there may not be some other way of representing the density operator in Eq. (13), say by

$$\tilde{\rho}_N = \int d\Omega_\psi \left(|\psi\rangle\langle\psi|\right)^{\otimes N}, \tag{20}$$

where the $|\psi\rangle$ represent a completely different set of states than the coherent states and $d\Omega_\psi$ represents a measure on that set. The answer to this question lies in the quantum de Finetti representation theorem [9,10].

Consider a source that sequentially introduces an *infinite* set of quantum systems, the first N of which are described by a density operator $\tilde{\rho}_N$. We shall make two requirements of this *sequence* of density operators. First, they should all be compatible in the sense that $\tilde{\rho}_N$ can be derived from $\tilde{\rho}_{N+1}$ by performing a partial trace over the Hilbert space of the $(N+1)$'th system. And second, for each N, $\tilde{\rho}_N$ should have the property that interchanging any two of the systems will not change the joint probability distribution for the outcomes of measurements on any of the individuals—that is to say, the density operator $\tilde{\rho}_N$ should remain invariant under a permutation of the systems it describes.

The quantum de Finetti representation theorem [9,10] specifies that—with these assumptions alone—the quantum state of any N systems from such a source can *necessarily* be written in the form

$$\tilde{\rho}_N = \int \mathrm{d}\rho P(\rho)\rho^{\otimes N} \ , \tag{21}$$

where $P(\rho)$ is a probability distribution over the density operators and $\mathrm{d}\rho$ is a measure on that space. Most importantly for the considerations here, this representation is *unique* up the behavior of $P(\rho)$ on a set of measure zero.

The meaning of this result in the present context is manifest: To the extent that one believes that a laser beam can be chopped into equal pieces and rearranged without affect to one's experiments—that is, that the beam is *stationary*—the representation in (13) is the only possibility of the form Eq. (20). That is to say, one can always act as if the temporal modes of a propagating laser beam are all in the same fixed but *unknown* coherent state. There are no other quantum states that will fit this bill.

Indeed, contemplate performing a set of measurements on the individual systems emanating from such a source as above. As the data accumulates, the probability distribution $P(\rho)$ in (21) should be updated according to standard Bayesian rules after the acquisition of that information [13]. Specifically, if measurements on K systems yield results D_K, then the state of additional systems is constructed as in Eq. (21), but using an updated probability distribution given by

$$P(\rho|D_K) = \frac{P(D_K|\rho)P(\rho)}{P(D_K)} \ . \tag{22}$$

Here $P(D_K|\rho)$ is the probability to obtain the measurement results D_K, given the state $\rho^{\otimes K}$ for the K measured systems, and

$$P(D_K) = \int P(D_K|\rho)\, P(\rho)\, d\rho \tag{23}$$

is the unconditional probability for the measurement results.

For a sufficiently informative set of measurements—namely a set of measurements whose eigenspaces span the whole linear vector space of operators over the initial Hilbert space—as K becomes large, the updated probability $P(\rho|D_K)$ becomes highly peaked on a particular state ρ_{D_K} dictated by the measurement results, regardless of the prior probability $P(\rho)$, as long as $P(\rho)$ is nonzero in a neighborhood of ρ_{D_K}. In other words, the measurement results essentially collapse the original mixed state to a new one in which any number M of additional systems are assigned the product state $\rho_{D_K}^{\otimes M}$, i.e.,

$$\int P(\rho|D_K)\, \rho^{\otimes M}\, d\rho \quad \longrightarrow \quad \rho_{D_K}^{\otimes M} \tag{24}$$

for K sufficiently large.

Comparing the state of a propagating laser field (13) with the general form (21) we see that a *complete* set of measurements on part of the light emanating from the laser will reduce the quantum state of the rest of the light to a pure state. But most importantly, this pure state will be a *coherent state*—there is no way to make it a number state; there is no way to make it a squeezed state or any other kind of exotic state. In this sense, the coherent states play a privileged role in the description of laser light [17].

It is true that standard optics experiments have not yet featured such complete measurements. For instance, a complete set for the case at hand would be a measurement of amplitude and absolute phase. However, recent developments [18] may make it possible to compare the phase of an optical light beam directly to the phase of a microwave field. Using this technique the only further measurement required for a complete measurement is a measurement of the absolute phase of the microwave field, which is possible electronically. This measurement would create an optical coherent state from a standard laser source for the first time. But as we will show in the next section, such a measurement does not even need to be performed for many applications.

E. Beamsplitters

In Section II B we constructed a set of *temporal* modes and used those to write down the quantum state of a propagating laser field. Alternatively, one may use *spatial* modes. In particular, we may imagine dividing a laser beam into an arbitrary number, N, of spatially different pieces by using beamsplitters. In general, the action of a beamsplitter with reflection and transmission coefficients r, t on a coherent state is given by

$$|\alpha e^{i\phi}\rangle|0\rangle \mapsto |r\alpha e^{i\phi}\rangle|t\alpha e^{i\phi}\rangle, \tag{25}$$

where the notation indicates that both input and output of a beamsplitter consist of two modes, and where here one input mode is in the vacuum state. Now given one discrete mode in a coherent state we may indeed apply the transformation (25) multiple times to obtain a state of the form (13). This method may seem simpler than that used in Section II B to derive (13) but it has two disadvantages. First, one still has to justify the discrete mode one started with. Indeed, one choice would be to choose a particular *temporal* mode. Second, once one has chosen a beamsplitter setup that divides the laser into N spatial modes, each carrying the same amount of light, one can no longer extend the set to $N+1$ spatial modes without changing the amplitudes of the original N modes. The de Finetti theorem could, therefore, not be applied to a sequence of quantum states so constructed. Of course, to extend a set of *temporal* modes by a further temporal one is easy: One just waits a little longer.

III. MIXED-STATE DESCRIPTION OF OPTICAL EXPERIMENTS

Let us now describe a few typical optical experiments using (13) for a proper description of the quantum state of a laser. This is sufficient for our purposes but we also discuss how the effects of phase diffusion modify the description.

A. Phase Measurement for Independent Lasers

Mølmer in [6] showed that the detection of a phase difference between two (independent) light beams need not imply that there is a well-defined phase difference before the measurement. In particular, he showed that for light emanating from two cavities whose fields are initially in number states (whose phase is completely random), the standard setup to measure phase will indeed find a stable phase difference (though the value of this phase will be random and different from experiment to experiment). Within one experiment, it takes just a few (about three) photon detections [6] to settle on a particular value of the phase difference, after which the counting rates of the detectors remain consistent with that initial phase difference. In other words, the standard phase measurement acts almost like a perfect von Neumann measurement; the measurement will produce an eigenvalue of the corresponding observable and the state after the measurement can be described by an eigenstate of the measured variable.

Generalizing this observation to continuously pumped CW lasers leads to the following simple description. Initially we have two independent laser beams A and B whose joint quantum state is described by

$$\tilde{\rho}_{2N} = \int \frac{\mathrm{d}\varphi_A}{2\pi} \big(|\alpha_A e^{i\varphi_A}\rangle\langle\alpha_A e^{i\varphi_A}|\big)^{\otimes N} \otimes \int \frac{\mathrm{d}\varphi_B}{2\pi} \big(|\alpha_B e^{i\varphi_B}\rangle\langle\alpha_B e^{i\varphi_B}|\big)^{\otimes N} \tag{26}$$

if we divide each laser beam into N packages of constant duration. If the first package of each beam is used to measure a phase difference then the state of the rest of the light beams will be reduced to

$$\tilde{\rho}_{2N-2} = \int \frac{\mathrm{d}\varphi_A}{2\pi} \big(|\alpha_A e^{i\varphi_A}\rangle\langle\alpha_A e^{i\varphi_A}|\big)^{\otimes (N-1)} \otimes \big(|\alpha_B e^{i(\phi_0+\varphi_A)}\rangle\langle\alpha_B e^{i(\phi_0+\varphi_A)}|\big)^{\otimes (N-1)}, \tag{27}$$

where we assumed the outcome of the phase measurement was ϕ_0 and approximated the measurement to be sharp. The state (27) has the property that a subsequent measurement of the phase difference will reproduce the value ϕ_0: This is a kind of "phase-locking without phase." Note this would certainly not be the case if the quantum state of a laser were a product of identical mixed states of the form $(\rho_{|\alpha|})^{\otimes N}$. Also note that in the number-state basis such properties are hard to understand.

For a phase-diffusing laser, (17) shows that measurements on adjacent parts of the laser beam will give approximately the same value for phase, whereas measurements on parts that are further apart than the diffusion time will give random results.

B. Coherent Excitation of a Two-Level Atom

In many experiments atomic coherence has been demonstrated: Superpositions of atomic states, degenerate or nondegenerate, have been supposedly created by using lasers. But how can one create such a coherent superposition if the laser field apparently is not coherent? Let us describe a typical experiment. If a laser would produce a coherent state $|\alpha\rangle$ it could be used to create an equal superposition of ground and excited states of a two-level atom by applying a $\pi/2$ pulse. In the limit of a large coherent-state amplitude the laser field would not become entangled with the atom [15] and one can write for the process taking place [16]

$$|\alpha\rangle \otimes |g\rangle \mapsto |\alpha\rangle \otimes (|g\rangle + |e\rangle)/\sqrt{2} \tag{28}$$

with $|g,e\rangle$ the atomic ground and excited states. However, taking into account the actual state of a laser field we have

$$\int \frac{\mathrm{d}\varphi}{2\pi} \big(|\alpha e^{i\varphi}\rangle\langle\alpha e^{i\varphi}|\big)^{\otimes N} \otimes \Pi_g \hookrightarrow \int \frac{\mathrm{d}\varphi}{2\pi} \big(|\alpha e^{i\varphi}\rangle\langle\alpha e^{i\varphi}|\big)^{\otimes N} \otimes \Pi_\phi\ , \tag{29}$$

where Π_g is the projector onto the ground state $|g\rangle$ and

$$\Pi_\phi = \frac{1}{2}\big(|g\rangle + e^{i\phi}|e\rangle\big)\big(\langle g| + e^{-i\phi}\langle e|\big). \tag{30}$$

The "phase" of the atomic superposition is equal to the "phase" of the coherent state. If we would trace out the laser field the resulting atomic density matrix would be completely mixed, an equal mixture of ground and excited states. However, in every experiment exploiting atomic coherence it is the same laser (or one that has been "phase-locked" to the same laser) that is used to perform a measurement on the atom. For instance, one applies another $\pi/2$ pulse to take the atom to the excited state and subsequently measures the atomic population in ground and excited states. But this works just as well with a laser in the mixed state since the *overall* process is independent of the phase of the laser. That is, by applying the second $\pi/2$ pulse one gets

$$\int \frac{\mathrm{d}\varphi}{2\pi} \big(|\alpha e^{i\varphi}\rangle\langle\alpha e^{i\varphi}|\big)^{\otimes N} \otimes \Pi_\phi \hookrightarrow \int \frac{\mathrm{d}\varphi}{2\pi} \big(|\alpha e^{i\varphi}\rangle\langle\alpha e^{i\varphi}|\big)^{\otimes N} \otimes \Pi_e, \tag{31}$$

and the probability to detect the atom in the state $|e\rangle$ is unity, whereas this probability would be 1/2 if uncorrelated laser beams would have been used.

If one considers this same experiment in the number-state basis the fact that an incoherent mixture of excited and ground states is transformed into a pure excited state is rather miraculous and seems to depend on very special correlations between the original laser pulse and the measurement pulse [19].

C. Production and Detection of Squeezed States

A squeezed state may be produced with the help of a nonlinear process described by an interaction Hamiltonian

$$H_I = \chi[a^\dagger b^2 + b^{\dagger 2} a], \tag{32}$$

where a and b are annihilation operators of single modes inside an optical resonator with frequencies ω_0 and $\omega_0/2$, respectively, and χ is proportional to the second-order nonlinearity $\chi^{(2)}$ of the nonlinear medium placed inside the cavity. Pump photons at frequency ω_0 can be down-converted to pairs of photons of frequency $\omega_0/2$. The resulting quantum state of the downconverted photons may display nonclassical two-photon correlations. If the pump field is in a coherent state and the mode b is initially in the vacuum state, then the state produced is a squeezed vacuum. On the other hand, if the pump field is in a mixed state diagonal in the number state basis, then the resulting state of mode b is also diagonal in the number state basis, since the interaction H_I preserves the number operator $N = 2a^\dagger a + b^\dagger b$.

As a consequence, if we write $|S_\alpha(\varphi)\rangle$ for a squeezed vacuum state produced by a laser in a coherent state with amplitude $\alpha e^{i\varphi}$ with α real, then the mixed state

$$\int \frac{\mathrm{d}\varphi}{2\pi} |S_\alpha(\varphi)\rangle\langle S_\alpha(\varphi)| \tag{33}$$

is not a squeezed state and does not display any nonclassical features, as this state too is a mixed state diagonal in the photon number state basis. (Similarly, in a simple picture the degrading effect of phase diffusion on squeezing can be understood by considering a mixture of squeezed states with phases drawn from a Gaussian probability distribution. A more detailed discussion is given in [14].) Yet, the state that is actually obtained in an experiment is of the form

$$\int \frac{\mathrm{d}\varphi}{2\pi} \left(|S_\alpha(\varphi)\rangle\langle S_\alpha(\varphi)|\right)^{\otimes M} \otimes \left(|\alpha e^{i\varphi}\rangle\langle \alpha e^{i\varphi}|\right)^{\otimes(N-M)}, \tag{34}$$

where we assumed that M light packets traversed the nonlinear medium and $N-M$ did not. Note that tracing out the unsqueezed part leaves a residue of de Finetti form: What remains can be viewed as a mixture of identical copies of some unknown squeezed state. Furthermore, using the exchangeability of that density operator, the quantum de Finetti theorem implies this expansion is unique. A complete tomographic measurement on some of those copies will reduce the quantum state of the remaining copies to a simple tensor product of squeezed states.

The state (34) will display nonclassical correlations between the squeezed mode(s) and the remaining laser light. In fact, those correlations do not depend on the value of φ, and, therefore, are the same as those that would be measured if one had a coherent state. The distinction between properties of a state like (33) and the same state but with the correlations with the laser light included, becomes more pronounced when we consider a two-mode squeezed state.

D. Production of Two-Mode Squeezed States

A two-mode squeezed state can be generated by splitting two squeezed states on a 50-50 beamsplitter. The resulting state of the two output ports is an entangled state. Denote a two-mode squeezed state generated from a coherent state with amplitude $\alpha e^{i\varphi}$ by $|T_\alpha^{AB}(\varphi)\rangle$, where the superscripts A, B refer to two distinct modes located in different laboratories, say Alice's and Bob's. As shown in [7], the state

$$\int \frac{\mathrm{d}\varphi}{2\pi} |T_\alpha^{AB}(\varphi)\rangle\langle T_\alpha^{AB}(\varphi)| \tag{35}$$

contains no entanglement between A and B: Instead, it simply denotes classical correlation between photon numbers for the two modes.

Now, however, suppose that some of the remaining laser light is supplied to Alice (as for instance for the purpose of producing a local oscillator [20]). The overall quantum state between Alice and Bob will then be of the form

$$\int \frac{\mathrm{d}\varphi}{2\pi} |T_\alpha^{AB}(\varphi)\rangle\langle T_\alpha^{AB}(\varphi)|^{\otimes M/2} \otimes \left(|\alpha_{A'} e^{i\varphi}\rangle\langle \alpha_{A'} e^{i\varphi}|\right)^{\otimes N}, \tag{36}$$

where A' indicates the further modes in Alice's possession, and where we assumed all M (assumed an even number) squeezed copies from the state (34) were split on the beamsplitter but the unsqueezed part was not. Far from being an unentangled state, this state has every bit as much entanglement as if the laser were actually a pure coherent source. It is just that the entanglement is in the form of *distillable entanglement* [21].

To see this, contemplate Alice doing a complete measurement on the extra laser light in her lab. With it, she will reduce the quantum state of modes A, B to a true two-mode squeezed state. Since these measurements are local (all measurements are performed on Alice's modes A'), it follows there must be distillable entanglement between Alice's and Bob's modes. Although the claim in [7] that the state (35) can be produced locally by Alice and Bob is quite correct, the state (36) is entangled and cannot be so produced.

For a phase diffusing laser the distillable entanglement in a state like (36) will be slightly less than the entanglement present in the corresponding two-mode squeezed state, because a measurement by Alice on part of her light would reduce the state of modes A and B to a slightly noisy version of a two-mode squeezed state. In the same simple picture as used before, that noise can be understood as arising from the randomness of the phase ϕ of the state $|T_\alpha^{AB}(\phi)\rangle$ caused by phase diffusion.

E. Teleportation with Continuous Variables

Since the state (36) does possess entanglement teleportation of continuous variables is possible even with lasers in mixed states. The actual procedure used in [20] required, as was noted in [7], both Alice and Bob to use some of the light of the same laser that generated the two-mode squeezed state to perform homodyne detection. The fact that Bob shares laser light with Alice does not imply however, that they share an active quantum channel over and above their original entanglement. One can imagine that all the light in Alice and Bob's possession (both the shared two-mode squeezed state and the light for their local oscillators) was sent to them *before* any actual teleportation takes place.

This may, if one wishes, be considered an additional shared resource that had not been made explicit before, but in that regard it is fairly innocent. As pointed out in [22], such a shared resource is necessary for any teleportation protocol, irrespective of its physical implementation. For teleportation with continuous variables, Alice and Bob need to share a synchronized clock; sharing some of the laser light is a practical way of implementing this. In fact, from a technological point of view laser light gives us the best possible clock [23]. (Similarly, in order to test Bell inequality violations with continuous-variable entangled states the local oscillator field necessary for the required measurements must be transported as well, see for example [24].) Thus, in contrast to [7], we do not consider the presence of this resource, which acts as a phase reference, as invalidating teleportation. An independent party, Victor, who would like to verify Alice and Bob's teleportation skills, could use his own laser but has to "phase-lock" it (in the sense of Section A) with Alice's laser. After all, Alice's claim is only that she can teleport a quantum state of a particular mode: Victor is free to choose the state to be teleported, but not the Hilbert space.

Finally, a crucial point is that the teleportation procedure as a whole does not depend on the precise value of the absolute phase φ. Therefore, for teleportation to succeed, Alice does not even have to do an absolute phase measurement to actually distill the entanglement present in the state (36). Teleportation can be achieved without knowing the imagined "unknown" phase φ arising in any PEF. Note in particular that Alice and Bob can teleport a quantum state handed to them by the independent third party Victor even if he is able to generate a pure coherent state or a pure entangled state. This is because the phases of both input and output state are compared to one and the same phase reference.

IV. CONCLUSIONS

In conclusion, viewing the laser beam of a CW laser as a sequence of N quantum systems led us to the following result: The quantum state of an ideal laser beam is a mixture of N copies of identical pure *coherent* states. By the quantum de Finetti representation theorem, the coherent states play a unique role in that regard. Such a state is very different from N copies of identical mixed states (be they mixtures of number states or of coherent states). One consequence is that appropriate measurements performed on part of a laser beam will reduce the quantum state of the rest of the laser beam to a pure coherent state (or a slightly noisy version thereof if one considers a realistic laser). Such measurements may in fact be possible with present-day technology [18], and thus an optical coherent state may in fact be generated. No sophisticated measurement on the laser medium [6] need be contemplated to carry this out.

Most importantly, this description allows us to properly assess quantum communication protocols that rely on lasers. In particular we found that teleportation with continuous variables is possible with conventional lasers without actually having to reduce the quantum state of a laser to a coherent state.

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[1] M. Sargent, M. O. Scully, and W. E. Lamb, *Laser Physics*, Addison-Wesley, Reading (1974).

[2] D. F. Walls and G. J. Milburn, *Quantum Optics*, Springer-Verlag, Berlin 1994.

[3] C. M. Caves and C. A. Fuchs, "Quantum Information: How Much Information in a State Vector?," in *The Dilemma of Einstein, Podolsky and Rosen — 60 Years Later*, edited by A. Mann and M. Revzen [Ann. Isr. Phys. Soc. **12**, 226 (1996).
[4] S. L. Braunstein, C. M. Caves, R. Jozsa, N. Linden, S. Popescu, and R. Schack, Phys. Rev. Lett. **83**, 1054 (1999).
[5] P. Kok and S. L. Braunstein, Phys. Rev. A **61**, 042304 (2000).
[6] K. Mølmer, Phys. Rev. A **55**, 3195 (1997); J. Mod. Optics **44**, 1937 (1997).
[7] T. Rudolph and B. C. Sanders, Phys. Rev. Lett. **87**, 077903 (2001).
[8] M. J. Collett and C. W. Gardiner, Phys. Rev. A **30**, 1386 (1984).
[9] R. L. Hudson and G. R. Moody, Z. Wahrs. **33**, 343 (1976).
[10] C. M. Caves, C. A. Fuchs, and R. Schack, `quant-ph/0104088`.
[11] W. Vogel and D.-G. Welsch, *Lectures on Quantum Optics*, (Akademie verlag GmbH, Berlin, 1994), Chapter 8.
[12] K. J. Blow, R. Loudon, S. J. D. Phoenix, and T. J. Shepherd, Phys. Rev. A **42**, 4102 (1990).
[13] R. Schack, T. A. Brun, and C. M. Caves, Phys. Rev. A **64**, 6401305 (2001).
[14] P. D. Drummond and M. D. Reid, Phys. Rev. A **37**, 1806 (1988).
[15] S. J. van Enk and H. J. Kimble, `quant-ph/0107088`, to be published in QIC.
[16] For this particular problem of two $\pi/2$ pulses one may, alternatively, define two nonoverlapping functions $\Phi_1(t)$ and $\Phi_2(t)$ corresponding to the two envelopes of the pulses that can then be extended to form a complete set of functions $\{\Phi_i(t)\}$. The two temporal modes thus defined are then, to a good approximation, the only two nonempty modes. One finds, as before, a fixed phase relationship between the two states. See [15] for more details.
[17] It is noteworthy that a very different approach based on so-called robust unravelings of Master equations describing open quantum systems also indicates a privileged role for the coherent states, namely that of the pure states that remain closest to pure during evolution. See, for example, J. Gea-Banacloche, Found. Phys. **28**, 531 (1998); H. M. Wiseman, Phys. Rev. A **47**, 5180 (1993).
[18] D. J. Jones, S. A. Diddams, J. K. Ranka, A. Stentz, R. S. Windeler, J. L. Hall, and S. T. Cundiff, Science **288**, 635 (2000).
[19] J. Gea-Banacloche, private communication.
[20] A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science **282**, 706 (1998).
[21] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[22] S. J. van Enk, J. Mod. Optics **48**, 2049 (2001).
[23] H. M. Wiseman, `quant-ph/0104004`.
[24] M. D. Reid, Phys. Rev. Lett. **84**, 2765 (2000) and references therein.