

# Photons in polychromatic rotating modes

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We propose a quantum theory of rotating light beams and study some of its properties. Such beams are polychromatic and have either a slowly rotating polarization or a slowly rotating transverse mode pattern. We show there are, for both cases, three different natural types of modes that qualify as rotating, one of which is a new type not previously considered. We discuss differences between these three types of rotating modes on the one hand and non-rotating modes as viewed from a rotating frame of reference on the other. We present various examples illustrating the possible use of rotating photons, mostly for quantum information processing purposes. We introduce in this context a rotating version of the two-photon singlet state.

## I. INTRODUCTION

Several recent papers describe “rotating beams of light” [1, 2, 3]. Such beams may have, for example, a rotating linear or slightly elliptical polarization, and should not be confused with circularly polarized light. At a fixed instant of time the direction of the (linear or elliptical) polarization vector rotates as a function of the propagation coordinate, and in a fixed plane perpendicular to the propagation direction the polarization rotates as a function of time. In a different type of rotating light beams it is the transverse intensity pattern rather than the polarization direction that is rotating. Such rotating beams of light may be produced by passing stationary beams through rotating optical elements, such as astigmatic lenses and half-wave plates [4, 5, 6, 7]. The rotational frequency  $\Omega$  of the optical elements (and hence of the light beams) will typically be very slow compared to the optical frequency  $\omega$  of the light beam.

The theory discussed in the cited papers [1, 2, 3] is classical, although it is noted that some effects are more conveniently understood in terms of photons. Some interesting paradoxes and even some contradictions are mentioned in [2], but the contradictions are not resolved there. The contradictions arise when one expects properties of rotating light beams to equal those of nonrotating light beams as seen from a rotating frame of reference. Here we give a quantum description of rotating light beams. We show that there are in fact several different natural ways of defining “rotating photons”. Moreover, we show that a *different* type of rotating photons arises by applying a rotation operator to standard (nonrotating) quantized modes. The latter photons describe photons seen from a rotating frame. Carefully distinguishing these different types of rotating photons thus removes the contradictions mentioned above from Ref. [2].

We will be particularly interested in the angular momentum of rotating light beams. As we will show below,

it will be easier to calculate the average angular momentum in a quantum description than in the classical descriptions of [1, 2, 3]. For example, it is pointed out in Refs. [1, 3] that one should be careful when applying expressions for the angular momentum derived for *monochromatic* light beams [8, 9]. Indeed, rotating light beams are necessarily *polychromatic*. In the formalism we use here no such problems arise and the quantum formalism takes care of polychromatic modes automatically.

Naturally, most salient features of rotating photons can be described in terms of angular momentum of light, simply because angular momentum operators generate rotations in space [10]. This angular momentum can have a spin or an orbital nature [11]. We thus start out by defining a complete set of electromagnetic field modes as follows: we use monochromatic modes with definite values of both spin and orbital angular momentum in the  $z$  direction,  $S_z$  and  $L_z$ . The corresponding quantum numbers are denoted by  $\omega$  (for energy),  $m$  (for orbital angular momentum) and  $s = \pm 1$  (for spin angular momentum, or, more precisely, for helicity). The modes may be exact solutions of the Maxwell equations (Bessel modes [12]), or they may be exact solutions of the paraxial equation, for modes propagating in the  $z$  direction (Laguerre-Gaussian modes). We must assume for the exact Bessel modes, however, that they, too, are propagating mostly in the  $z$  direction. That is, we assume  $k_T \ll \omega/c$ , with  $k_T$  the magnitude of the *transverse* components of the wave vector. This condition is needed in order for  $L_z$  and  $S_z$  to be well-defined angular momenta with integer eigenvalues [13]. So we use paraxial modes in either case.

A rotating mode or photon is then defined as an (almost) equal superposition of two opposite angular momenta  $l$  and  $-l$ , with different frequencies  $\omega \pm l\Omega$ . Photons with a rotating polarization are superpositions of two opposite spin angular momenta; photons with a rotating transverse mode pattern are superpositions of opposite orbital angular momenta.

Besides the three quantum numbers mentioned so far, there is a fourth quantum number necessary to fully specify an arbitrary mode. This fourth quantum number describes the remaining transverse spatial degree of freedom. It could be the number of zeros  $n_T$  in the transverse mode pattern of a Laguerre-Gaussian mode or the transverse momentum  $\hbar k_T$  of a Bessel mode [13]. For our purposes we do not have to specify the transverse degrees of freedom any further. We thus assume that the fourth quantum number is fixed, so that we can use a simplified notation and denote the modes by the indices  $(\omega, m, s)$ .

The rest of the paper is organized as follows. In the next Section we will introduce notation and define more precisely modes with definite amounts of spin and orbital angular momenta. Those modes are monochromatic. In Section III we discuss how to define, in general, polychromatic modes. These are used to define quantized modes describing rotating photons in Sections IV and V. We will use the Heisenberg picture, as it allows the most direct comparison of the fields with the classical case treated before in the literature. As it turns out we can define at least three different types of rotating modes, and we will discuss the angular momentum of these various types of rotating photons. In Section IV we define rotating photons that have no angular momentum on average, in Section V we define two types of rotating photons with angular momentum, either parallel or antiparallel to the propagation direction. Measurements at the single-photon level of rotating modes are discussed at the end of Section IV. In Section VI we define modes that correspond to nonrotating modes as seen from a rotating frame of reference, and we indicate the differences from the rotating modes of Sections IV and V. In Section VII we consider some applications of single-photon or two-photon states of rotating modes, in particular, the use of rotating photons as a means of encoding quantum information. We summarize in Section VIII.

## II. PRELIMINARIES

For a given mode, the negative-frequency component of the (dimensionless) classical electric field can be written in cylindrical coordinates as

$$\vec{F}_{\omega,m,s}(z, \rho, \phi, t) = \exp(im\phi) \exp(-i\omega t) F(\rho, z) \vec{e}_s, \quad (1)$$

which is valid for the free field. The polarization vectors are  $\vec{e}_{\pm} = (\vec{e}_x \pm i\vec{e}_y)/\sqrt{2}$ . There are other nonzero components of the electric field but they are small in the paraxial approximation. We focus our attention on the main component (1).

In Eq. (1) we left the dependence of the field on  $\rho$  and  $z$  unspecified. The precise form of  $F$  depends on whether we use exact or paraxial modes. For example, in the case of the exact Bessel modes we have [13]

$$F(\rho, z) \propto J_{|m|}(k_T \rho) \exp(ik_z z), \quad (2)$$

where  $k_z$  is the longitudinal component of the wave vector,  $k_z^2 = k^2 - k_T^2$ , and  $J_m$  is the  $m$ th-order Bessel function.  $F$  depends on the quantum numbers  $k_T$  and  $\omega$  in this case, but not on  $s$  and only on the absolute value  $|m|$ . For paraxial modes we have the more involved expressions for the Laguerre-Gaussian modes [14]. Also in that case,  $F$  depends on the quantum numbers  $\omega$ , the absolute value  $|m|$ , and the quantum number  $n_T$ , but not on the polarization index. This observation plays an important role later on, when we define modes as superpositions of different modes that always have the same value of  $|m|$ .

The Bessel modes do not diffract, and the transverse intensity pattern is independent of  $z$ . There is a  $z$ -dependent phase factor, and by choosing it equal to  $\exp(ik_z(z-z_0))$  for some fixed  $z_0$  for *all* modes, we ensure that the mode functions in the plane  $z = z_0$  are independent of the frequency  $\omega$ , if we fix the remaining quantum numbers  $k_T, m, s$ . We will refer to the plane  $z = z_0$  as the reference plane. For Bessel modes each plane has the same intensity configuration. For paraxial modes, on the other hand, we do have to define a particular location of the reference plane, and we choose the same value  $z_0$  for *all* paraxial modes. Such modes form a complete set of (paraxial) modes. Usually, the reference plane will be chosen as the focal plane, where the wavefronts are flat.

## III. TIME-DEPENDENT MODES

### A. Field operators and mode functions

Here we consider the theory of quantized modes, with as starting point the mode functions (1). We use the Heisenberg picture, so that operators rather than states depend on time. The (time-dependent) creation and annihilation operators for modes with quantum numbers  $\omega, m, s$  are denoted by  $\hat{a}_{\omega,m,s}^\dagger(t)$  and  $\hat{a}_{\omega,m,s}(t)$ . (Recall that we leave out the transverse quantum numbers). The frequency is a continuous variable, and the mode operators are assumed to obey the standard bosonic commutation rules  $[\hat{a}_{\omega',m,s}(t), \hat{a}_{\omega,m,s}^\dagger(t)] = \delta(\omega - \omega')$ . Likewise, the single-photon states  $|\omega, m, s\rangle = \hat{a}_{\omega,m,s}^\dagger|\text{vac}\rangle$  are delta-function normalized. For a free field, the time dependence of the mode operators is simply

$$\hat{a}_{\omega,m,s}(t) = \exp(-i\omega t) \hat{a}_{\omega,m,s}(0). \quad (3)$$

We need the electric field operator from the relevant modes. In the present paper, we can restrict ourselves to the paraxial modes propagating in the positive  $z$  direction. The contribution of a single paraxial mode to the positive-frequency part of the electric field operator is

$$\hat{E}_{\omega,m,s}^{(+)}(t) = \sqrt{\frac{\hbar\omega}{4\pi\epsilon_0 c}} \vec{F}_{\omega,m,s} \hat{a}_{\omega,m,s}(t) =: \vec{E}_{\omega,m,s} \hat{a}_{\omega,m,s}(t), \quad (4)$$

with  $\vec{F}$  given by (1). Here we indicate operators by adorning them with hats. The normalization factor pro-

portional to the square root of the frequency  $\omega$  ensures the proper form of the Hamiltonian, in the form

$$\hat{H} = \sum_{m,s} \int d\omega \hbar\omega \hat{a}_{\omega,m,s}^\dagger \hat{a}_{\omega,m,s} . \quad (5)$$

This normalization is based on the assumption that the mode functions  $\vec{F}_{\omega,m,s}$  are normalized in each transverse plane, as is common for paraxial modes. This means that  $\int \rho d\rho d\phi |\vec{F}_{\omega,m,s}|^2 = 1$ . The presence of the  $\sqrt{\omega}$  term is responsible for the existence of various different types of rotating photons, as we will see below.

For later use we display the expression for contribution of a mode to the positive-frequency part of the vector potential in the Coulomb gauge as

$$\hat{A}_{\omega,m,s}^{(+)}(t) = -i\sqrt{\frac{\hbar}{4\pi\epsilon_0\omega C}} \vec{F}_{\omega,m,s} \hat{a}_{\omega,m,s}(t) =: \vec{A}_{\omega,m,s} \hat{a}_{\omega,m,s}(t) . \quad (6)$$

Finally, we also give here the expression for the spin and orbital angular momentum operators that will play a crucial role in the rest of the paper:

$$\begin{aligned} \hat{S}_z &= \sum_{m,s} \int d\omega \hbar s \hat{a}_{\omega,m,s}^\dagger \hat{a}_{\omega,m,s} , \\ \hat{L}_z &= \sum_{m,s} \int d\omega \hbar m \hat{a}_{\omega,m,s}^\dagger \hat{a}_{\omega,m,s} . \end{aligned} \quad (7)$$

The operator for the total angular momentum is denoted as  $\hat{J}_z = \hat{S}_z + \hat{L}_z$ .

## B. Unitary transformations of modes

From now on we shall use for simplicity a generic subscript  $i$  to indicate the full set of quantum numbers  $\omega$ ,  $m$ ,  $s$  and the remaining transverse mode number, so that the summation over  $i$  represents a summation over  $m$  and  $s$  and an integration over  $\omega$ . In this notation, the operators for the positive-frequency part of the electric field and the vector potential are denoted as

$$\vec{\hat{E}}(t) = \sum_i \vec{E}_i \hat{a}_i(t) , \quad \vec{\hat{A}}(t) = \sum_i \vec{A}_i \hat{a}_i(t) . \quad (8)$$

Given a complete set of field modes  $\vec{E}_i$  and corresponding mode operators  $\hat{a}_i$ , we can define a different complete set of (orthonormal) modes and mode operators in a general way. For this purpose we transform the field modes  $\vec{E}_i$  as

$$\vec{E}'_i = \sum_j U_{ij} \vec{E}_j , \quad (9)$$

where  $U_{ij}$  is a unitary matrix. As will become obvious in the next section, the specific transformations we will consider couple only a limited number of modes, so that the summation in (9) extends over a few discrete indices

$j$ . A key feature of the transformation is that it couples modes with different frequencies, so that the primed modes are not monochromatic. In order that the electric field operator can be expanded as

$$\vec{\hat{E}}(t) = \sum_i \vec{E}'_i \hat{a}'_i(t) , \quad (10)$$

we must transform the set of annihilation operators by

$$\hat{a}'_i(t) = \sum_j U_{ij}^* \hat{a}_j(t) . \quad (11)$$

The unitarity of  $U$  ensures that the new modes are still orthogonal and that the new creation and annihilation operators still satisfy the correct equal-time commutation relations. Due to the unitarity of  $U$ , the inverse expansion of (11) is given by the transpose matrix, and we find for time zero

$$\hat{a}_j(0) = \sum_i U_{ij} \hat{a}'_i(0) . \quad (12)$$

For the *free* field the time dependence of the electric-field operator can be explicitly taken into account by incorporating the time dependence in the mode functions. Substituting the inverse expansion (12) into Eq. (8) gives the resulting expression

$$\begin{aligned} \vec{\hat{E}}(t) &= \sum_{i,j} \vec{E}_j \exp(-i\omega_j t) U_{ij} \hat{a}'_i(0) \\ &=: \sum_i \vec{E}'_i(t) \hat{a}'_i(0) , \end{aligned} \quad (13)$$

where the last line defines time-dependent and possibly non-monochromatic mode functions  $\vec{E}'_i(t)$ . By this transformation we can easily make a connection between the quantum theory of rotating photons and the classical theory of rotating light beams. On the basis of the new modes, the electric-field operator  $\vec{\hat{E}}(t)$  is now expressed either as an expansion (10) with time-dependent mode operators, or as an expansion (13) in time-dependent non-monochromatic mode functions. It is noteworthy that although the summations are the same, the summands are not. Only when the transformation does not couple modes with different frequencies are the two expansions (10) and (13) the same term by term.

## IV. ROTATION WITHOUT ANGULAR MOMENTUM

To describe modes rotating at a frequency  $\Omega$  around the  $z$  axis, we start with monochromatic modes at some frequency  $\omega$ . We take equal superpositions of *two* fields with opposite angular momenta (either spin or orbital) and shift their frequencies by opposite amounts, proportional to the angular momentum. In fact, we can choose to have a rotating transverse intensity pattern by shifting

the frequency proportional to the orbital angular momentum, or a rotating polarization by shifting in proportion to the spin angular momentum. It is convenient to consider these cases separately.

### A. Rotating polarization

As an example of Eq. (11), we define new mode operators

$$\hat{b}_{\pm} := (\hat{a}_{\omega+\Omega s, m, s} \pm \hat{a}_{\omega-\Omega s, m, -s})/\sqrt{2}. \quad (14)$$

This transformation is, by Eq. (9), accompanied by the mode definitions

$$\vec{E}_{\pm}^b := (\vec{E}_{\omega+\Omega s, m, s} \pm \vec{E}_{\omega-\Omega s, m, -s})/\sqrt{2}. \quad (15)$$

For ease of notation we indicate the various transformed modes by different letters, rather than by primes. The new modes  $b$  are described by new quantum numbers. For instance,  $\omega$  is a nominal frequency now, and no longer the eigenfrequency of the mode. Indeed, the new mode has no eigenfrequency anymore. The index  $\pm$  replaces the bi-valued polarization index  $s$ . The transverse quantum number, not displayed explicitly, stays the same, and so does  $m$ , the orbital angular momentum. The rotation frequency  $\Omega$  is *not* an additional quantum number, but rather a parameter labeling the complete set of modes defined by (15). Indeed, whereas fixing a particular quantum number always restricts the set of modes to some smaller subset, fixing  $\Omega$  still leaves one with a complete set of modes. On the other hand, we could try to consider  $\Omega$  a quantum number if we *fix* a particular value of  $\omega = \omega_0$ . This would not be a natural choice, though, especially when  $\Omega \ll \omega$ . Moreover, modes with frequencies larger than  $2\omega_0$  would not be included [34].

The reason for calling these new modes “rotating” is as follows: the extra time-dependent terms in the electric field operator rotating at a frequency  $\pm\Omega$  due to the change in frequency can be absorbed into the polarization part. For instance, take  $s = 1$  and consider the reference plane  $z = z_0$ . In that plane the mode functions  $F(\rho, z_0)$  for the 2 modes appearing in the definition for  $b$  are identical, by construction. The transformation (13) for the electric-field operators and the  $b_{\pm}$  modes takes the form

$$\vec{E}_{+}^b(t)\hat{b}_{+}(0) + \vec{E}_{-}^b(t)\hat{b}_{-}(0) = \vec{E}_{+}^b\hat{b}_{+}(t) + \vec{E}_{-}^b\hat{b}_{-}(t). \quad (16)$$

In the reference plane these modes can be written as

$$\begin{aligned} \vec{E}_{\pm}^b(t)\hat{b}_{\pm} &= \sqrt{\frac{\hbar\omega}{4\pi\epsilon_0 c}} \exp(-i\omega t) \exp(im\phi) F(\rho, z_0) \hat{b}_{\pm} \\ &\times [\cos\theta \vec{e}_{+} \exp(-i\Omega t) \pm \sin\theta \vec{e}_{-} \exp(i\Omega t)], \end{aligned} \quad (17)$$

where we define

$$\begin{aligned} \cos\theta &= \sqrt{\frac{\omega + \Delta}{2\omega}} \\ \sin\theta &= \sqrt{\frac{\omega - \Delta}{2\omega}}, \end{aligned} \quad (18)$$

with  $\Delta = \Omega$  the frequency shift. The last line in Eq. (17) is the time-dependent polarization vector

$$\begin{aligned} \vec{e}(t) &= A_{\pm}(\vec{e}_x \cos\Omega t + \vec{e}_y \sin\Omega t) \\ &+ iA_{\mp}(-\vec{e}_x \sin\Omega t + \vec{e}_y \cos\Omega t), \end{aligned} \quad (19)$$

where

$$A_{\pm} = \frac{\cos\theta \pm \sin\theta}{\sqrt{2}}. \quad (20)$$

Since typically we will have  $\Omega \ll \omega$  (so that  $A_{+}$  is very close to 1, and  $A_{-}$  is close to 0), the rotating polarization is almost linear for both the  $b_{+}$  mode and the  $b_{-}$  mode. Both modes  $b_{\pm}$  describe an elliptical polarization whose axes rotate in the same direction at a frequency  $\Omega$  around the  $z$  axis. In fact, it is easy to verify that a time shift by  $\tau_1 = \pi/(2\Omega)$  transforms the  $b_{+}$  mode into the  $b_{-}$  mode and *vice versa*. More precisely,

$$\vec{E}_{\pm}^b(t + \pi/(2\Omega)) = -i \exp(-i\pi\omega/(2\Omega)) \vec{E}_{\mp}^b(t). \quad (21)$$

Nevertheless these two modes are distinct, and the mode operators  $\hat{b}_{+}$  and  $\hat{b}_{-}^{\dagger}$  commute.

So far we considered the field in the reference plane only. If we go outside the reference plane  $z = z_0$  then for Bessel modes we still find the polarization is rotating in the same way in each plane  $z = \text{constant}$ . For solutions of the paraxial equations (Gaussian beams), however, the two components of the field at different frequencies diffract in slightly different ways. Nevertheless, as long as we do not stray too far from the reference plane, the polarization still rotates in more or less the same way. Similar conclusions will hold for all modes discussed below. We will always display the field in the focal plane  $z = z_0$ , and it should be kept in mind that our descriptions are in general meant to apply near the reference plane.

### B. Rotating transverse mode pattern

In a similar way we may define new modes by

$$\hat{c}_{\pm} := (\hat{a}_{\omega+\Omega m, m, s} \pm \hat{a}_{\omega-\Omega m, -m, s})/\sqrt{2}. \quad (22)$$

This transformation is, by Eqs. (9) and (11), accompanied by the mode definitions

$$\vec{E}_{\pm}^c := (\vec{E}_{\omega+\Omega m, m, s} \pm \vec{E}_{\omega-\Omega m, -m, s})/\sqrt{2}. \quad (23)$$

These modes obey a relation similar as Eq. (16), with  $b$  replaced by  $c$ . Now the extra time-dependent terms in the electric field operator rotating at a frequency  $\pm\Omega m$  due to the change in frequency can be absorbed into the azimuthal part. Just as before, let us take  $s = 1$  and consider the reference plane  $z = z_0$ . The electric-field operators for the  $c_{\pm}$  modes in that plane can (using (13))

be written as

$$\begin{aligned} \vec{E}_{\pm}^c(t)\hat{c}_{\pm} &= \sqrt{\frac{\hbar\omega}{4\pi\epsilon_0 c}} \exp(-i\omega t) F(\rho, z_0) \vec{e}_{\pm} \hat{c}_{\pm} \\ &\times [\cos\theta \exp(im(\phi - \Omega t)) \\ &\pm \sin\theta \exp(-im(\phi - \Omega t))] \end{aligned} \quad (24)$$

with the same definition (18) of the angle  $\theta$  as before, except that now  $\Delta = m\Omega$ . Clearly, the time-dependent field has a transverse mode pattern that rotates with a frequency  $\Omega$  around the  $z$  axis. Again, for both modes  $c_{\pm}$  the direction of the rotation is the same. And just as for the polarization case, a time shift interchanges the modes  $c_{\pm}$ . Now the time shift that accomplishes this is a shift by  $\tau_m = \pi/(2m\Omega)$ , so that

$$\vec{E}_{\pm}^c(t + \pi/(2m\Omega)) = -i \exp(-i\pi\omega/(2m\Omega)) \vec{E}_{\mp}^c(t). \quad (25)$$

Finally, we note that the quantum numbers of the modes  $c_{\pm}$  are different than those of the original modes  $a$ : in particular, instead of the quantum number  $m$  we have now both  $m$  and  $-m$ , while keeping  $s$  and  $\omega$  (although the latter is no longer the eigenfrequency of the modes).

### C. Rotating polarization and mode pattern

There is nothing to prevent us from defining modes where both the transverse mode profile *and* the polarization are rotating at a frequency  $\Omega$ . We just define

$$\hat{d}_{\pm} := (\hat{a}_{\omega+\Omega(m+s),m,s} \pm \hat{a}_{\omega-\Omega(m+s),-m,-s})/\sqrt{2}. \quad (26)$$

We can even define modes where the polarization is rotating at a different frequency than the transverse mode pattern,

$$\hat{e}_{\pm} := (\hat{a}_{\omega+\Omega m+\Omega' s,m,s} \pm \hat{a}_{\omega-\Omega m-\Omega' s,-m,-s})/\sqrt{2}. \quad (27)$$

Since all these redefinitions are unitary, the corresponding electric field amplitudes will, by construction, still be valid normalized solutions of the appropriate wave equations.

In order to see what it means to have the polarization and the transverse mode profile rotating at different frequencies, let us consider one explicit example. Suppose we define

$$\hat{f}_{\pm} = (\hat{a}_{\omega+\Omega,+1,-1} \pm \hat{a}_{\omega-\Omega,-1,+1})/\sqrt{2}. \quad (28)$$

Then this mode can be viewed as having a transverse mode pattern that rotates in the positive direction at frequency  $\Omega$ . Indeed, for any fixed linear polarization component, its field distribution rotates in the positive direction. On the other hand, the same mode can also be seen as having a polarization vector that rotates in the *negative* direction. That is, if we fix any point in the reference plane, then the local polarization vector rotates in the negative direction. The reason is simple:

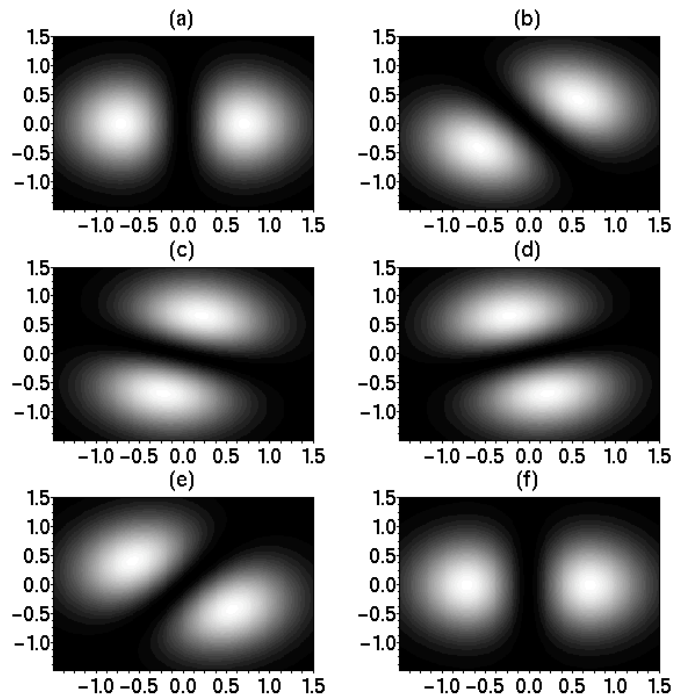


FIG. 1: Transverse intensity profile of the  $x$  component of the field corresponding to the mode  $f_+$ , as a function of  $x$  and  $y$  (in arbitrary units) in the focal plane  $z = 0$ . Snapshots are shown at 6 different times: for figures (a)–(f) we have  $\Omega t = n\pi/5, n = 0 \dots 5$ , respectively. Here  $\Omega > 0$  and the sense of rotation of the intensity profile is positive.

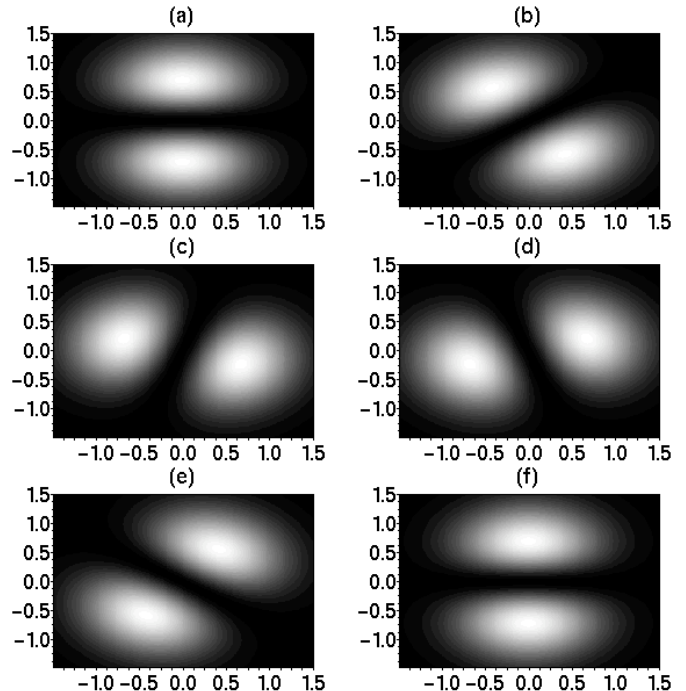


FIG. 2: Same as for Fig. 1, but for the  $y$  component.

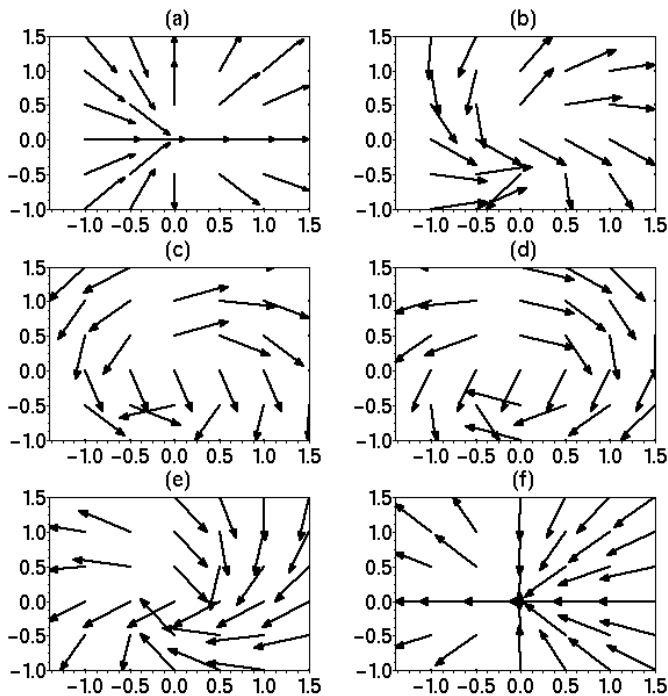


FIG. 3: The (linear) polarization vector of the field corresponding to the mode  $f_+$ . Snapshots are shown at the same instants as in Figs. 1 and 2. Note the direction of rotation of the polarization vector is opposite (negative) to that of the transverse intensity patterns of Figs. 1 and 2, although we still have  $\Omega > 0$ .

the extra time-dependent phase factors  $\exp(\pm i\Omega t)$  can be absorbed either in the polarization vector or in the transverse mode pattern, and the choice is arbitrary, of course. We illustrate this behavior in Figs. 1–3. We plot snapshots at different times of the intensity profiles for the  $x$  and  $y$  components of the field in Figs. 1 and 2, respectively. The polarization direction is plotted in Fig. 3 at the same instants of time.

#### D. Rotating single photons produced by rotating mode inverters

Now consider a single photon in any one of the modes we have defined so far. For instance, consider a state of the form

$$|1\rangle_b = \hat{b}_+^\dagger |\text{vac}\rangle, \quad (29)$$

where  $|\text{vac}\rangle$  denotes the vacuum state, with all modes unoccupied by photons. The coherence properties of a single-photon state are characterized by the complex matrix element of the electric-field operator

$$\langle \text{vac} | \hat{E}(\vec{r}, t) | 1 \rangle_b = \vec{E}_+^b(\vec{r}, t). \quad (30)$$

This quantity, which is the quantum analog of the classical electric field, is the detection amplitude function of

the photon. It determines the second-order coherence of a one-photon field [15].

This photon has an average spin angular momentum of zero, although its polarization is rotating at a frequency  $\Omega$ , according to (19). The simple reason is that the photon is in an *equal* superposition of spin angular momentum eigenstates with eigenvalues  $+\hbar$  and  $-\hbar$ . There is no contradiction in having a rotating polarization and yet zero spin, as there is no simple *linear* relation between polarization and spin angular momentum. The expectation value of the electric-field amplitude is in fact always zero for any single-photon state, but the spin angular momentum is determined, in both the classical case and the quantum case, by a bilinear function of the field amplitudes; the spin is nonzero for any single photon with a definite polarization that is not linear. The average energy  $\langle E \rangle$  of the  $b$  photon is  $\hbar\omega$ , again because it is in an *equal* superposition of energy eigenstates with energies  $\hbar\omega \pm \hbar\Omega$ .

Similar conclusions hold for the other modes,  $c$ ,  $d$ ,  $e$ , and  $f$  defined in the previous subsections. That is, for each such mode the average energy of a single photon is  $\hbar\omega$ . For the mode  $c$ , the orbital angular momentum vanishes, while for the other modes the total angular momentum is zero, even though these modes obviously display rotation.

In [3] it was shown that rotating photons are generated by rotating mode inverters. Suppose one has an optical element that “inverts” the polarization vector of a light beam according to

$$\vec{e}_s \mapsto \vec{e}_{-s} \quad (31)$$

for  $s = \pm 1$ . This is the effect of a half-wave plate. Then a plate that rotates at an angular frequency  $\Omega/2$  will generate a mode with a polarization vector rotating at a frequency  $\Omega$ . The doubling of the rotation frequency can be understood by noting that in a rotating frame the mapping (31) becomes

$$\vec{e}_s \exp(is\Omega t/2) \mapsto \vec{e}_{-s} \exp(-is\Omega t/2). \quad (32)$$

The quantum equivalent of this mapping is

$$\hat{a}_{\omega, m, s} \mapsto \hat{a}_{\omega - s\Omega, m, -s}. \quad (33)$$

The linear superposition  $(\hat{a}_{\omega, m, 1} + \hat{a}_{\omega, m, -1})/\sqrt{2}$  of mode operators corresponds to a mode with linear polarization. This linear superposition is mapped onto

$$(\hat{a}_{\omega, m, 1} + \hat{a}_{\omega, m, -1})/\sqrt{2} \mapsto \hat{b}_+, \quad (34)$$

so that a linearly polarized single-photon state is mapped onto the state  $|1\rangle_b$ . This shows that a rotating half-wave plate with linear polarized photons as input generates  $b_+$  photons as output.

We can also take as input a linear superposition of modes with orbital mode indices  $m$  and  $-m$ , onto a mode converter rotating at an angular frequency  $\Omega/2$ . The

corresponding classical mapping is

$$\vec{E}_{\omega,m,s} \exp(im\Omega t/2) \mapsto \vec{E}_{\omega-m\Omega,-m,s} \exp(-im\Omega t/2). \quad (35)$$

There is, however, an ambiguity here: a mode converter will have an input plane  $z = z_i$  and a different output plane  $z = z_o$ . Since between those planes modes with different frequencies will diffract differently, a rotating mode converter works properly only when  $z_i$  and  $z_o$  are sufficiently close for those diffraction effects to be negligible. Assuming this is the case, the quantum equivalent of the mapping by a rotating mode converter is

$$\hat{a}_{\omega,m,s} \mapsto \hat{a}_{\omega-m\Omega,-m,s}. \quad (36)$$

The mode operator corresponding to the superposition of two modes with opposite orbital angular momentum  $\pm m$  is mapped as

$$(\hat{a}_{\omega,m,s} + \hat{a}_{\omega,-m,s})/\sqrt{2} \mapsto \hat{c}_+. \quad (37)$$

A single photon in this superposition mode is therefore converted into the single-photon state  $|1\rangle_c$ , with the single-photon wave function  $\vec{E}_+^c(\vec{r}, t)$ . Again, since this photon is in an equal superposition of two states with orbital angular momentum  $\pm \hbar m$ , its average orbital angular momentum is zero, even though the mode pattern is rotating. This agrees with the conclusions for a classical rotating field created by a rotating mode inverter [3].

### E. Measurements on single rotating photons

Polarization measurements on single photons are certainly possible. If one wishes to distinguish, say,  $x$  and  $y$  polarized photons, all one needs is a polarizing beam splitter oriented such that  $x$  and  $y$  polarized photons exit at different output ports, and a subsequent photodetection event performs a (destructive) projective measurement. Similar projective measurements distinguishing different orbital angular momentum eigenstates of single photons are possible as well [16]. In that case, too, one can construct a sorting device that splits an incoming stream of photons into different output channels with different (orthogonal) angular momentum states. A subsequent photodetection finalizes the projective measurement.

Can one do the same for rotating modes? That is, can one construct a similar sorting device for distinguishing, e.g., the modes  $b_+$  and  $b_-$  even for single photons? Using the preceding subsection it is easy to see one certainly can. Namely, one simply sends the input photon through a rotating half wave plate, after which a polarization measurement achieves the desired projective measurement. In order to distinguish modes  $c_+$  and  $c_-$  at the single-photon level one first sends the photon through a rotating mode converter, and subsequently one needs a sorting device that distinguishes between different Hermite-Gaussian modes. Fortunately, such devices, working at the single-photon level, exist as well [17].

## V. ROTATION WITH ANGULAR MOMENTUM

There is an alternative way of defining mode transformations. If we consider the expression for the rotating polarization of the mode of Eq. (17), then we see extra prefactors  $\sin \theta$  and  $\cos \theta$  appearing because of the (quantum) normalization factor proportional to  $\sqrt{\hbar\omega}$ . Similar factors appear in Eq. (24) for the rotating transverse mode pattern, for the same reason. In order to compensate for those prefactors we replace the mode operators (14) by the definition

$$\begin{aligned} \hat{g}_+ &= \sin \theta \hat{a}_{\omega+\Omega s, m, s} + \cos \theta \hat{a}_{\omega-\Omega s, m, -s}, \\ \hat{g}_- &= \cos \theta \hat{a}_{\omega+\Omega s, m, s} - \sin \theta \hat{a}_{\omega-\Omega s, m, -s}. \end{aligned} \quad (38)$$

The compensation of the factor  $\sqrt{\omega}$  works only for the '++' mode of the pair of modes (38), but the companion '--' modes are necessary to make the redefinition unitary.

For the '++' modes we get instead of (17) the expression for the electric-field operator

$$\begin{aligned} \vec{E}_+^g(t) \hat{g}_+ &= \sqrt{\frac{\hbar\omega}{\epsilon_0 V}} \exp(-i\omega t) \sin \theta \cos \theta \exp(im\phi) F(\rho, z_0) \\ &\times \hat{g}_+ [\vec{e}_+ \exp(-i\Omega t) + \vec{e}_- \exp(i\Omega t)]. \end{aligned} \quad (39)$$

The last line now describes a rotating *linear* polarization

$$\vec{e}(t) = \vec{e}_x \cos \Omega t + \vec{e}_y \sin \Omega t. \quad (40)$$

A single photon from this mode now does possess a nonzero average spin angular momentum, equal to

$$\langle \hat{S}_z \rangle = (\sin^2 \theta - \cos^2 \theta) \hbar = -\hbar \Omega / \omega. \quad (41)$$

This result is exact, not perturbative, even though typically we do have  $\Omega/\omega \ll 1$ . The photon is in an unbalanced superposition of states with angular momentum  $+\hbar$  with relative weight  $\sin^2 \theta$  and with angular momentum  $-\hbar$  with weight  $\cos^2 \theta$ . Similarly, the energy of a  $g_+$  photon is

$$\langle E \rangle = \hbar\omega - \hbar\Omega^2/\omega. \quad (42)$$

Of course we can define analogous modes with rotating transverse mode patterns, while also compensating for the extra factors  $\sin \theta$  and  $\cos \theta$  in Eq. (24), by defining

$$\begin{aligned} \hat{h}_+ &= \sin \theta \hat{a}_{\omega+\Omega m, m, s} + \cos \theta \hat{a}_{\omega-\Omega m, -m, s}, \\ \hat{h}_- &= \cos \theta \hat{a}_{\omega+\Omega m, m, s} - \sin \theta \hat{a}_{\omega-\Omega m, -m, s}. \end{aligned} \quad (43)$$

It is again only the  $h_+$  modes for which the  $\theta$ -dependent prefactors in the electric-field amplitude cancel. That is, the electric field of such a mode rotates around the  $z$  axis with the same shape as in the non-rotating case  $\Omega = 0$ . A single photon in the  $h_+$  mode has an orbital angular momentum equal to

$$\langle \hat{L}_z \rangle = -\hbar m^2 \Omega / \omega, \quad (44)$$

and the energy is  $\langle E \rangle = \hbar\omega - \hbar m^2 \Omega^2 / \omega$ . Thus for both modes  $g_+$  and  $h_+$  the angular momentum is, perhaps counter-intuitively, negative for a mode rotating in the positive direction around the  $z$  axis.

Interestingly, the  $'-'$  modes that we were forced to define by requiring unitarity also have a nice property: for these modes it is the  $\omega$ -dependent prefactors in the expression for the vector potential that cancel, rather than in the expression for the electric field. Thus the  $g_-$  mode describes a vector potential whose direction rotates uniformly and without changing length around the  $z$  axis. Similarly, for the  $h_-$  modes the transverse mode pattern of the vector potential rotates around the  $z$  axis without changing shape. For single photons in the  $'-'$  modes we find now that the angular momentum has the opposite value as for the  $'+'$  modes. Thus we have

$$\langle \hat{S}_z \rangle = \hbar\Omega / \omega, \quad (45)$$

for a  $g_-$  photon and

$$\langle \hat{L}_z \rangle = \hbar m^2 \Omega / \omega, \quad (46)$$

for a  $h_-$  photon. The energy of the photons is  $\langle E \rangle = \hbar\omega + \hbar\Omega^2 / \omega$  and  $\langle E \rangle = \hbar\omega + \hbar m^2 \Omega^2 / \omega$ , respectively. So here the energy per photon is higher than  $\hbar\omega$ , while for the  $'+'$  modes it was lower by the same amount. As far as the authors are aware, the  $'-'$  modes have not been discussed before.

Of course, these other types of rotating photons can be generated with rotating mode inverters by taking different superpositions as input.

## VI. PHOTONS AS SEEN FROM A ROTATING FRAME

The modes we have constructed so far are “rotating modes”. The field modes satisfy the Maxwell equations or the paraxial equations, and the mode operators satisfy the Heisenberg equations of motion. It is useful to compare those modes to the modes we get by applying a rotation operator of the form

$$\hat{R}(t) = \exp(i\hat{J}_z \Omega t) \quad (47)$$

to non-rotating modes. The transformed mode operators

$$\hat{a}'(t) = \hat{R}^\dagger \hat{a}_{\omega,m,s}(t) \hat{R} = \exp(i\Omega(m+s)t) \hat{a}_{\omega,m,s}(t) \quad (48)$$

no longer satisfy the correct Heisenberg equations of motion for a free field, because the unitary rotation operator depends on time. Instead the mode operators and the corresponding field operators describe modes as seen from a rotating frame, rotating at an angular frequency  $\Omega$  around the  $z$  axis. One may easily confuse operators like  $\exp(i\Omega(m+s)t) \hat{a}_{\omega,m,s}(t)$  with the similar operators  $\hat{a}_{\omega-\Omega(m+s),m,s}(t)$ . Their time dependence is the same, but since the corresponding mode functions have different frequencies, they satisfy different equations of motion, and display different diffraction behavior.

On the other hand, the fact that a rotating beam of light can be described by taking superpositions of modes with different values of angular momentum and shifting the frequency in proportion to the angular momentum can be explained by this very analogy. The frequency shift can be seen as a rotational version of the Doppler shift [6].

Alternatively, the frequency shift proportional to angular momentum can be seen as a time-dependent manifestation of a geometric phase [5], with the time derivative of the phase equaling the frequency shift. For a rotating polarization this shift arises from the Pancharatnam phase [18], for a rotating transverse mode pattern it arises from the similar “orbital” geometric phase associated with mode transformations [19]. The latter geometric phase was measured recently in its time-independent form [20] by using the mode converter from [21].

It may be that there is a deeper connection between angular momentum of light and the various geometric phases of light: according to Refs. [22] the geometric phase arises only when angular momentum is exchanged, and this was confirmed in special cases in [19, 23]. A recent experiment [24] indicates that this connection between angular momentum exchange and the occurrence of a geometric phase may be more general.

## VII. EXAMPLES

We give here some examples of the use of rotating photons, mostly for quantum-information processing purposes. We do not claim rotating photons are superior than any other type of photons in this context, they just provide an interesting alternative.

We first discuss *wave-packets* of rotating photons.

### A. Wave-packets of rotating photons

The modes defined so far are of infinite spatial and temporal extent but are nevertheless fine for most theoretical purposes. In practice a more useful definition of photons is in terms of wavepackets that are of finite extent. They are easily defined in the usual way: consider a state of the form

$$|1\rangle_F := \int d\omega F(\omega) \hat{g}_+^\dagger(\omega) |\text{vac}\rangle, \quad (49)$$

where  $F(\omega)$  is a normalized function

$$\int d\omega |F(\omega)|^2 = \int dt |\tilde{F}(t)|^2 = 1, \quad (50)$$

with  $\tilde{F}(t)$  the Fourier transform

$$\tilde{F}(t) = \frac{1}{\sqrt{2\pi}} \int d\omega F(\omega) \exp(-i\omega t). \quad (51)$$



The corresponding electric field is then determined by

$$\langle \text{vac} | \hat{\vec{E}}(\vec{r}, t) | 1 \rangle_F = \int d\omega F(\omega) \vec{E}_{+, \omega}^g(t). \quad (52)$$

This electric field is proportional to

$$\vec{F}(t) [\vec{e}_x \cos \Omega t + \vec{e}_y \sin \Omega t]. \quad (53)$$

If we assume that  $F(\omega)$  has a finite width in frequency space, then the probability of detecting such a rotating photon is nonzero only in a finite time interval at each given position, and nonzero in a finite spatial interval at any given time. The rotating character of a wave-packet will be visible only if its time duration is sufficiently long. Roughly speaking, if the duration  $\Delta t > 2\pi/\Omega$ , then a full rotation of the polarization vector over  $2\pi$  is included in the wave-packet. In frequency space, this means the frequency spread should be at most of order  $\Omega$ . We will investigate how such a wave-packet interacts with a single atom in later in this Section.

## B. Entanglement between rotating modes

A *rotating* equivalent of the well-known singlet state may be defined as

$$|RS\rangle := \frac{\hat{g}_+^\dagger \otimes \hat{g}_-^\dagger - \hat{g}_-^\dagger \otimes \hat{g}_+^\dagger}{\sqrt{2}} |\text{vac}\rangle. \quad (54)$$

Here, in a tensor product the first operator is meant to refer to a mode located in  $A$ , the second to a mode located in a different location  $B$ . This state  $|RS\rangle$  then is entangled between modes (or locations)  $A$  and  $B$ . For example, this two-photon state is anticorrelated with respect to (average) spin angular momentum: if one detects one photon in the  $g_+$  state, then it has spin angular momentum equal to  $-\hbar\Omega/\omega$ . The other photon in the other location is then necessarily in the  $g_-$  state with the opposite average spin angular momentum.

But we can rewrite this same state in various different forms. For example, we also have

$$|RS\rangle = \frac{\hat{b}_+^\dagger \otimes \hat{b}_-^\dagger - \hat{b}_-^\dagger \otimes \hat{b}_+^\dagger}{\sqrt{2}} |\text{vac}\rangle. \quad (55)$$

So if we measure whether one photon is in the  $b_+$  or  $b_-$  mode, then the other photon will be found in the other mode. Here both photons have zero spin angular momentum on average.

Finally, we can also write

$$|RS\rangle = \frac{\hat{a}_{\omega+\Omega,+}^\dagger \otimes \hat{a}_{\omega-\Omega,-}^\dagger - \hat{a}_{\omega-\Omega,-}^\dagger \otimes \hat{a}_{\omega+\Omega,+}^\dagger}{\sqrt{2}} |\text{vac}\rangle \quad (56)$$

in terms of our original mode operators. So if one photon is found to have an energy  $\hbar(\omega + \Omega)$  then the other one must have an energy  $\hbar(\omega - \Omega)$ . And if one photon has

spin angular momentum  $+\hbar$  then the other has  $-\hbar$ . The rotating singlet state  $|RS\rangle$  thus has in common with the standard nonrotating singlet state (at a single frequency  $\omega$ ) that the total angular momentum is zero, and that the total energy is  $2\hbar\omega$ .

On the other hand, if in the state  $|RS\rangle$  one photon is found rotating, then the other is rotating in the *same* direction, even when the angular momenta are in fact opposite. Moreover, when trying to violate Bell inequalities with the state  $|RS\rangle$  the standard polarization measurements in fixed bases will not do. Instead one needs polarization measurements in co-rotating bases.

Similar conclusions hold for the orbital equivalent of  $|RS\rangle$ , defined by

$$\begin{aligned} |RSO\rangle &:= \frac{\hat{c}_+^\dagger \otimes \hat{c}_-^\dagger - \hat{c}_-^\dagger \otimes \hat{c}_+^\dagger}{\sqrt{2}} |\text{vac}\rangle \\ &= \frac{\hat{h}_+^\dagger \otimes \hat{h}_-^\dagger - \hat{h}_-^\dagger \otimes \hat{h}_+^\dagger}{\sqrt{2}} |\text{vac}\rangle. \end{aligned} \quad (57)$$

## C. Quantum cryptography with rotating photons

For quantum key distribution, in particular for the BB84 protocol [25, 26], one needs two mutually unbiased bases. One could (in theory rather than in practice) use single photons and encode information in polarization. One orthodox choice could be to use single photons in the modes  $a_{\omega,+}$  and  $a_{\omega,-}$  as one pair of orthogonal states (basis), corresponding to circular polarization, and another pair of modes corresponding to linear polarization, e.g.  $(a_{\omega,+} \pm a_{\omega,-})/\sqrt{2}$ .

In the present context, we could of course contemplate using one pair of orthogonal *rotating* modes  $b_\pm$  [with zero average spin angular momentum] and another pair of nonrotating modes  $a_{\omega+\Omega,+}$  and  $a_{\omega-\Omega,-}$ , such that the overlaps between states from different bases are equal to 50%. In the latter set the two basis states can be distinguished by a frequency measurement but also by a polarization measurement. On the other hand, the former two basis states can be distinguished by neither of those measurements, and instead one could use a polarization measurement in combination with a precise timing measurement ( $b_+$  is a time-shifted version of  $b_-$ ). More precisely, one basis requires a frequency measurement with an accuracy better than  $\Omega$ , the other basis requires a timing measurement better than  $\pi/(4\Omega)$ , thus revealing the complementarity between the two bases. We could say that with the help of the (rotating) polarization degree of freedom, the conjugate variables used for this implementation of BB84 are time and frequency, rather than noncommuting spin measurements  $\sigma_z$  and  $\sigma_x$  in the orthodox implementation.

This implementation is related to, but different from, time-bin entangled photons for quantum key distribution [27]. In fact, the rotating version of the BB84 protocol is equivalent to an entanglement-based protocol that makes

use of the rotating singlet state  $|RS\rangle$  from the preceding subsection. The fact we can write the same state in different forms, namely (55) and (56), demonstrates this explicitly.

The generalization to the orbital equivalent is obvious.

#### D. Interference between rotating photons

A typical quantum effect is the appearance of a dip at zero delay in the number of coincidences of detector clicks behind a beam splitter as a function of delay between two input photons that enter the two input ports: the Hong-Ou-Mandel (HOM) dip [28]. If we consider the interference of two, say,  $b_+$  photons as a function of a time delay between them, then we find the standard HOM curve is modulated by an extra time-dependent factor. Namely, we have

$$b_+(t + \tau) = \cos \Omega\tau b_+(t) - \sin \Omega\tau b_-(t), \quad (58)$$

so that the modulation factor is simply

$$\cos^2 \Omega\tau = \frac{1}{2} + \frac{1}{2} \cos 2\Omega\tau. \quad (59)$$

This implies the HOM curve will display an oscillation at a frequency  $2\Omega$  as a result of the rotating character of the polarization. But, of course, it can also be viewed as a beat frequency between the  $\omega \pm \Omega$  components of the modes. Finally, we note the modulation factor is slightly different from the overlap of the two time-dependent polarizations,

$$|\vec{e}(t) \cdot \vec{e}^*(t + \tau)|^2 = \sin^4 \theta + \cos^4 \theta + 2 \sin^2 \theta \cos^2 \theta \cos(2\Omega\tau). \quad (60)$$

#### E. Single atoms and single rotating photons

In principle *any* degree of freedom of a photon can be used to encode information. One question that may arise in the present context is whether the information about the rotating polarization or rotating transverse mode pattern of a single photon can be stored or processed or transferred to a different information carrier. For that purpose let us consider here the interaction between a single photon and a single atom.

One can certainly store the *fixed* (non-rotating) polarization state of a photon in an atom by making use of the selection rules (or, equivalently, angular momentum conservation). Namely, starting off the atom in a particular magnetic sub level one can make the atom (in principle at least) absorb one photon by using stimulated Raman scattering by applying a, say,  $\hat{z}$  polarized laser beam. The atom will end up in a particular ground state picked out by the selection rules. For instance, if the initial atomic state is chosen to be  $|m_A = 0\rangle_A$  (we use a subscript  $A$  to indicate atomic degrees of freedom)

then by absorbing a  $\sigma^\pm$  photon the atom ends up in the state  $|m_A = \pm 1\rangle_A$  (assuming, of course, those levels exist in the atom). A photon with a general polarization  $\alpha\sigma^+ + \beta\sigma^-$  will put the atom in the corresponding coherent superposition  $\alpha|A + 1\rangle + \beta|-1\rangle_A$  in that case (where we use the symmetry of the Clebsch-Gordan coefficients for the transitions used).

But how can one store a *rotating* polarization? One could use one atom and create a rotating superposition  $\alpha \exp(-i\Omega t')|+1\rangle_A + \beta \exp(i\Omega t')|-1\rangle_A$  by lifting the energy degeneracy of the two  $m$ -levels by applying an external magnetic field. Here  $t' = t - t_0$  where  $t_0$  is determined by the time the magnetic field was switched on (and by how the magnetic field was turned on). This indeed is the quantum state of an atom whose magnetic moment is rotating in time. For this to work one needs *a priori* knowledge of the rotation frequency  $\Omega$ . This is, of course, consistent with the observation that  $\Omega$  is not really a quantum number, but a classical parameter labeling different complete sets of modes (note we need to know the value of  $\omega$  of the incoming photon as well, to match it with a resonant transition in the atom). In this way we can transfer one qubit from a single rotating photon to the rotating magnetic moment of a single atom. All that is needed is information on the precise timing (to within order  $\Omega^{-1}$ ) of the switching on of the magnetic field. Recall that the difference between a  $b_+$  photon and a  $b_-$  photon is just a time shift by  $\tau = \pi/(2\Omega)$ .

(We note there is in fact an experiment succeeding in mapping certain quantum states of light onto an atomic ensemble, using rotating spin states of atoms [29]. However, what is stored is a very different type of information, namely the amplitude of a coherent state. The input state of the light field is not rotating in the experiment of Ref. [29].)

Let us consider the process of an atom absorbing a photon with rotating polarization in more detail. We consider the case where the external degrees of freedom of the atom can be treated classically. An atom located at position  $z = z_A$  not too far from the focal plane  $z = z_0$  starting in the state  $|m_A = 0\rangle_A$  will turn into (using a subscript  $F$  to indicate the field degrees of freedom)

$$\begin{aligned} &|1\rangle_F \otimes |0\rangle_A \mapsto \\ &|0\rangle_F \otimes \int d\tau \tilde{F}(\tau - z'/c) \exp(-i\omega(\tau - z'/c)) \\ &[P(\omega + \Omega) \exp(-i\Omega(\tau - z'/c))|+1\rangle_A + \\ &P(\omega - \Omega) \exp(i\Omega(\tau - z'/c))|-1\rangle_A] \\ &+ \sqrt{P_0}|1\rangle_F \otimes |0\rangle_A, \end{aligned} \quad (61)$$

where  $z' = z_A - z_0$  is the position of the atom relative to the focal plane  $z = z_0$ . We included here a probability amplitude for the atom to absorb a photon at frequency  $\omega$  of the form  $P(\omega) \propto (\omega - \omega_A - i\Gamma)^{-1}$ , with  $\omega_A$  the appropriate 2-photon Raman resonance frequency and  $\Gamma$  an (effective Raman) decay width. The last line gives the term describing the case that the atom does not absorb the photon, with probability  $P_0$ . Assuming  $P_0 \ll 1$  we

see the atom will end up in a roughly equal superposition of  $|+1\rangle_A$  and  $|-1\rangle_A$ , as if excited by a photon with a linear polarization that is a weighted time average of the rotating polarization passing by. Also note that the position of the atom is irrelevant: the atom will see the same sequence of time-varying polarization vectors pass by, no matter where it is (provided it is still near the focal plane, so that diffraction effects can be neglected).

Now suppose the external degrees of freedom of the atom are treated quantum mechanically, and the atom starts out in a pure state of its center-of-mass motion. Because of energy and momentum conservation the atom ends up in a state that displays correlation (or even entanglement) between its external and internal degrees of freedom. This way information about  $\Omega$  can, in principle, be stored. If we write the transformation the atom undergoes in symbolic notation as

$$\begin{aligned} &|1\rangle_F \otimes |0\rangle_A \otimes |E\rangle_A \mapsto \\ &|0\rangle_F \otimes \int d\tau \tilde{F}(\tau - z'/c) \exp(-i\omega(\tau - z'/c)) \\ &[P(\omega + \Omega) \exp(-i\Omega(\tau - z'/c)) | +1\rangle_A \otimes |E + \Omega\rangle_A + \\ &P(\omega - \Omega) \exp(i\Omega(\tau - z'/c)) | -1\rangle_A \otimes |E - \Omega\rangle_A] \\ &+ \sqrt{P_0} |1\rangle_F \otimes |0\rangle_A \otimes |E\rangle_A, \end{aligned} \quad (62)$$

then the information about  $\Omega$  is stored in the atom's external state provided, for instance,  $\langle E - \Omega | E - \Omega' \rangle = 0$  for  $\Omega' \neq \Omega$ . After all, in that case a measurement on the external state of the atom can reveal the value of  $\Omega$ .

Under those same circumstances the internal and external states of the atom are entangled, since we also have  $\langle E - \Omega | E + \Omega \rangle = 0$ , with maximum entanglement if  $|F(\omega + \Omega)| = |F(\omega - \Omega)|$ . The more the two states  $|E - \Omega\rangle_A$  and  $|E + \Omega\rangle_A$  overlap, the less entangled internal and external states are, and the less information is stored about the value of  $\Omega$ . In the extreme case of perfect overlap, we are back to the classical case: for example, the atom is in a quasi-classical coherent state of its external motion, and the energy  $\hbar\Omega$  is much smaller than the average motional energy of the atom, so that the small shift in energy is not detectable, and the external and internal degrees of freedom remain uncorrelated.

For completeness, let us note that in order to transfer information encoded in a rotating transverse mode profile, one necessarily needs to use the external degrees of freedom. Indeed, whereas spin angular momentum is coupled to the internal (electronic) degrees of freedom of an atom, the orbital angular momentum couples to the center-of-mass motion, as was shown explicitly in Ref. [30]. Again, one needs *a priori* knowledge of the value of  $\Omega$ , but then one can, in principle at least if not in practice yet, transfer a bit of information encoded in, say, the  $c_{\pm}$  modes to an atom. This would, however, be a much more involved experiment than related experiments producing entanglement between modes of different orbital angular momentum [31, 32].

## VIII. SUMMARY

We developed a quantum theory of rotating photons for which either the (linear or slightly elliptical) polarization vector or the transverse mode pattern rotates slowly around the propagation ( $z$ ) direction. The rotational frequency  $\Omega$  is independent of the optical frequency  $\omega$ , and will typically be much smaller than  $\omega$ . We found that there are, in each case, *three* natural types of rotating photons: they can have spin angular momentum  $-\hbar\Omega/\omega$ , 0, or  $+\hbar\Omega/\omega$ , if the photon has a rotating polarization, and an orbital angular momentum  $-m\hbar\Omega/\omega$ , 0, or  $+m\hbar\Omega/\omega$  if the photon has a rotating transverse mode pattern composed of modes with orbital angular momenta  $\pm m\hbar$ . These three types of rotating photons correspond to modes with a rotating unchanging electric field vector, an *equal* superposition of opposite angular momenta, and a rotating unchanging vector potential, respectively. These photons should be distinguished from nonrotating photons as viewed from a frame rotating at  $-\Omega$  around the  $z$  axis. We also defined propagating rotating wavepackets of finite duration, giving a more realistic picture of what would be produced in an experiment.

We then considered some examples of single-photon and two-photon states illustrating properties of rotating photons. We defined a rotating version of the standard singlet state, an entangled state consisting of two photons with opposite angular momenta. A new aspect of the rotating version of the state is that, if the polarization of one photon is measured and found rotating in the positive direction around the propagation axis, the other photon's polarization necessarily rotates in the same direction. But if one photon's polarization in that same state is found not to rotate, then neither does the other. Thus whereas the angular momenta of the two photons are anti-correlated, the sense of rotation is correlated.

Rotating photons also allow one to use different conjugate variables for, e.g., quantum key distribution. In particular, instead of using nonorthogonal polarization states to encode information, time and frequency can be used as conjugate variables, with the help of a rotating polarization. That is, information can either be stored in the frequency of a single nonrotating photon or in the timing of a rotating single photon, but one cannot measure both properties at the same time.

We then verified whether information about the rotating character of a single photon can be transferred to the state of a single atom. We argued one can certainly create a rotating magnetic moment matching the rotating polarization of an incoming photon in a single atom, provided one has *a priori* knowledge of the value of  $\Omega$ : one just uses a magnetic field to cause a Zeeman shift equal to  $\hbar\Omega$  and a superposition of different Zeeman sublevels  $|m = \pm 1\rangle$  then creates a rotating magnetic moment, rotating at a frequency  $\Omega$ . The fact that one needs classical information about the rotational frequency is directly related to the fact that  $\Omega$  is *not* a quantum number, but rather a classical parameter that labels different complete sets of

(quantum) modes. With degenerate magnetic sublevels the internal state of an atom stores merely a time-average of the rotating polarization of an incoming photon.

We also considered how the external motion of an atom can store certain types of information encoded in a rotating photon. The value of  $\Omega$  can be stored, and, similarly, a transverse mode pattern can be imprinted on the center-of-mass motion of a single atom. Using the center-of-mass motion is perhaps not a very practical idea, but we considered this case for the sake of completeness.

Finally, it may be interesting to see if the concept of rotating photons can prove useful in the field of time and frequency standards. After all, the optical frequency  $\omega$  of photons exploited in such standards is complemented by a slowly rotating polarization vector, which may be seen as an extra slow hand of a fast clock. This does require that  $\omega$  and the rotational frequency  $\Omega$  be locked to each other by some process [33]. It seems unlikely such a process exists in nature.

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- [1] C.N. Alexeyev and M.A. Yavorsky, *J. Opt. A* **7**, 416 (2005).
- [2] A.Yu. Bekshaev, M.S. Soskin, and M.V. Vasnetsov, *Opt. Commun.* **249**, 367 (2005).
- [3] G. Nienhuis, *J. Phys. B* **39**, S529 (2006).
- [4] B.A. Garetz and S. Arnold, *Opt. Commun.* **31**, 1 (1979).
- [5] R. Simon, H.J. Kimble, and E.C.G. Sudarshan, *Phys. Rev. Lett.* **61**, 19 (1988).
- [6] G. Nienhuis, *Opt. Commun.* **132**, 8 (1996).
- [7] J. Courtial, K. Dholakia, D. A. Robertson, L. Allen, and M. J. Padgett, *Phys. Rev. Lett.* **80**, 3217 (1998).
- [8] S.J. van Enk and G. Nienhuis, *Opt. Commun.* **94**, 147 (1992).
- [9] M. Berry, *Proc. SPIE* **3487**, 6 (1998).
- [10] There are some subtle issues concerning angular momentum operators for light [and for any massless field] and what type of rotations they generate, see S.J. van Enk and G. Nienhuis, *Europhys. Lett.* **25** 497 (1994); D. Terno, *Phys. Rev. A* **67**, 014102 (2003).
- [11] L. Allen, M.J. Padgett, and M. Babiker, *Progress in Optics* **39**, 291 (1999).
- [12] J. Durnin, J.J. Miceli, and J.H. Eberly, *Phys. Rev. Lett.* **59**, 2612 (1987).
- [13] S.J. van Enk and G. Nienhuis, *J. Mod. Optics* **41**, 963 (1994).
- [14] L. Allen, M.W. Beijersbergen, R.J.C. Spreeuw, and J.P. Woerdman, *Phys. Rev. A* **45**, 8185 (1992).
- [15] L. Mandel, and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, UK, 1995).
- [16] J. Leach *et al.*, *Phys. Rev. Lett.* **88**, 257901 (2002).
- [17] M. Duparre *et al.*, *Proc. SPIE* **3291**, 104 (1998); see also F. Spedalieri [quant-ph/0409057](#).
- [18] M.V. Berry, *J. Mod. Optics* **34** 1401 (1987).
- [19] S.J. van Enk, *Opt. Commun.* **102**, 59 (1993).
- [20] E.J. Galvez, P.R. Crawford, H.I. Sztul, M.J. Pysher, P.J. Haglin, and R.E. Williams, *Phys. Rev. Lett.* **90**, 203901 (2003).
- [21] M.W. Beijersbergen, L. Allen, H.E.L.O. van der Veen, and J.P. Woerdman, *Opt. Comm.* **96**, 123 (1993).
- [22] S.C. Tiwari, *J. Mod. Optics* **39**, 1097 (1992); *ibidem* **51** 2297 (2004).
- [23] D. Banerjee, *Phys. Rev. E* **56**, 1129 (1997); *J. Opt. Soc. Am. B* **23**, 817 (2006).
- [24] E. J. Galvez and M. O'Connell, *Proc. SPIE* **5736**, 166 (2005).
- [25] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing*, Bangalore, India (IEEE, New York, 1984), 175.
- [26] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, *Rev. Mod. Phys.* **74**, 145 (2002).
- [27] I. Marcikic, H. de Riedmatten, W. Tittel, H. Zbinden, M. Legr, and N. Gisin, *Phys. Rev. Lett.* **93**, 180502 (2004).
- [28] C.K. Hong, Z.Y. Ou, and L. Mandel, *Phys. Rev. Lett.* **59**, 2044 (1987).
- [29] B. Julsgaard, J. Sherson, J. I. Cirac, J. Fiurasek, E. S. Polzik, *Nature* **432**, 482 (2004).
- [30] S.J. van Enk, *Quantum Optics* **6**, 445 (1994).
- [31] A. Mair, A. Vaziri, G. Weihs and A. Zeilinger, *Nature* **412**, 313 (2001).
- [32] S. S. R. Oemrawsingh, A. Aiello, E. R. Eliel, G. Nienhuis, and J. P. Woerdman, *Phys. Rev. Lett.* **92**, 217901 (2004).
- [33] Jun Ye, private communication.
- [34] On the flipside, our mode definition requires that  $\Omega < \omega$ , so there is always a range of frequencies  $\omega$  for which rotating modes at angular velocity  $\Omega$  cannot be defined.