Solutions Final 252H

10:15–12.15, Friday, March 21st, 2008

- **Some useful constants:**

  $g = 9.81\text{m/s}^2$ (gravitational acceleration on the surface of Earth)

  $c = 3.00 \times 10^8\text{m/s}$ (speed of light)

  $\hbar = h/(2\pi) = 1.05 \times 10^{-34}\text{J}\cdot\text{s}=6.58 \times 10^{-16}\text{eV}\cdot\text{s}$ (Planck's constant divided by 2π)

  $R_E = 6.38 \times 10^6\text{m}$ (Earth's radius)

  $m_e c^2 = 511\text{keV}$ (electron's rest energy)

  $m_p c^2 = 938\text{MeV}$ (proton's rest energy)

- **A useful formula:** For small $|\epsilon|$ we can approximate $(1+\epsilon)^p \approx 1+p\epsilon+p(p-1)\epsilon^2/2+\ldots$

1 [20 points] An atom emits a photon with wavelength $\lambda = 700\text{nm}$ while decaying from an excited state to the ground state.

(a) What is the energy of the photon in eV?

(b) The life time of the atom's excited state is $\tau = 7\text{ns}$. Estimate the uncertainty ΔE in the energy of the photon (in eV).

(c) Estimate, using your answer to (b), the uncertainty $\Delta\lambda$ in the wavelength of the photon.

Answers:

(a) Nothing much to do here: $E = hc/\lambda = 1.77\text{eV}$.

(b) Using the uncertainty relation for energy and time gives (ignoring factors of 2, and hence using only 1 digit)

$$\Delta E \approx \frac{\hbar}{\tau} = 1 \times 10^{-7}\text{eV}.$$

The shorter the life time, the more uncertain the energy.

(c) Since $E = hc/\lambda$ we have

$$\frac{\Delta E}{E} = \frac{\Delta\lambda}{\lambda}.$$

(Differentiate: $dE/d\lambda = -hc/\lambda^2 = -E/\lambda$, and use that both ΔE and $\Delta\lambda$ are positive.) Thus (again using just 1 digit):

$$\Delta\lambda = \frac{\Delta E}{E}\lambda = \frac{9.4 \times 10^{-8}}{1.77}700\text{nm} \approx 4 \times 10^{-5}\text{nm}\,(= 4 \times 10^{-14}\text{m}).$$

The shorter the life time, the more uncertain the wavelength.

2 [10 points] According to one of Kepler's laws the period T of the orbit of a planet orbiting the Sun and a particular measure a of its distance to the Sun obey the relation:

$$\frac{T^2}{a^3} = \mathrm{C},$$

where C is a constant, the same for all planets. Show that this relation holds for the special case of a circular orbit and *derive* an expression for C in terms of G and the mass M_S of the Sun. (For each planet the influence of the other planets is neglected.)

Answer: For a circular orbit with radius a we have the usual relation

$$\frac{mv^2}{a} = \frac{GmM_s}{a^2}.$$

We also have $T = 2\pi a/v$, so that

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM_s}.$$

3 [20 points] Suppose we accelerate an electron such that it ends up with a final total energy of $E_f = 700\mathrm{MeV}$.

(a) How close is the electron's final speed v (in our reference frame, of course) to c? That is, determine $1 - v/c$.

(b) Suppose the same electron collides with a positron (which has the same rest mass as the electron) that moves at the same speed v but in the opposite direction. Suppose the electron and positron annihilate each other and produce two photons. Write down the equations expressing conservation of the energy-momentum 4-vector $\tilde{p}$ and derive the energies (again, in our reference frame) of the two photons in eV.

Answers:

(a) We have $\gamma = E/E_0 = 700 \times 10^6/511 \times 10^3 \approx 1.37 \times 10^3$. Hence

$$1 - \beta \approx \frac{1}{2\gamma^2} = 2.66 \times 10^{-7}.$$

[This makes use of an approximation valid when β is very close to 1: $1/\gamma^2 = 1 - \beta^2 = (1+\beta)(1-\beta) \approx 2(1-\beta)$.]

(b) Choose the positive x direction as the direction of motion of the electron. Before the annihilation we have then

$$\tilde{p}_e = (E/c, p, 0, 0),$$

where p is the momentum of the electron in the (positive) x direction: $p = \gamma m_0 v \approx E/c$ (since v is very close to c). For the positron we have, similarly,

$$\tilde{p}_p = (E/c, -p, 0, 0),$$

with the same values for p and E. After the annihilation process we have 2 photons with 4-momenta:

$$\tilde{p}_k = (p_k, \vec{p}_k),$$

for $k = 1, 2$ where $p_k = |\vec{p}_k|$. Conservation of momentum gives $p_1 = p_2$, and conservation of energy then gives $p_1 = p_2 = E/c$ for both photons. The energy of the photons is thus $E = 700\text{MeV}$ each. This obvious answer also follows from symmetry arguments.

4 [10 points] Suppose an electron and a proton have the same momentum. Comparing the (quantum-mechanical, non-relativistic) matter waves of the electron and the proton, the proton's wave has

(a) a shorter wavelength and a greater frequency

(b) a longer wavelength and a greater frequency

(c) the same wavelength and a greater frequency

(d) the same wavelength and the same frequency

(e) a longer wavelength and a smaller frequency

(f) a shorter wavelength and a smaller frequency

(g) the same wavelength and a smaller frequency

Answer: Since $\lambda = h/p$ the wavelengths are the same, and since $K = p^2/(2m)$ the (non-relativistic!) kinetic energy of the proton is smaller (as $m_p > m_e$), and so its frequency $\nu = K/h$ is smaller, too. So answer (g) is correct.

5 [20 points] A hoop, a sphere and a cylinder—all with the same mass and the same radius—roll (without slipping) down an incline.

(I) Rank them according to the **total kinetic energy** with which they arrive at the bottom, from highest to lowest:

(a) sphere, cylinder, hoop, (b) hoop, cylinder, sphere, (c) all the same

(II) Rank them according to the **angular speed** ω with which they arrive at the bottom, from highest to lowest:

(a) sphere, cylinder, hoop, (b) hoop, cylinder, sphere, (c) all the same

(III) Rank them according to the **angular momentum** (relative to the center of mass) with which they arrive at the bottom, from highest to lowest:

(a) sphere, cylinder, hoop, (b) hoop, cylinder, sphere, (c) all the same

Answers:

(I) Gravity does the same amount of work one each, the friction force does no work (for rolling without slipping), so the total kinetic energy must be the same for all: answer (c) is correct. In fact, we must have then:

$$\frac{1}{2}m(\omega R)^2 + \frac{1}{2}I\omega^2 = mgh.$$

(II) The rotational speed of the hoop must be the smallest, as it has the largest moment of inertia I (and the cylinder is next), given that energies are all the same: answer (a) is correct. In fact, we have

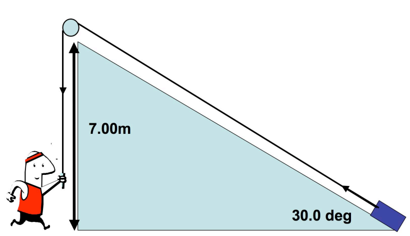
$$\omega = \sqrt{\frac{2mgh}{mR^2 + I}}.$$

(III) The friction force on the hoop must be the largest as it ends up with the smallest translational speed. Thus the torque ($\tau = F_f R$, which is due to the friction force *only*: gravity exerts no torque relative to the center of mass!) on the hoop is the largest, so it ends up with the largest angular momentum (and the cylinder is next): answer (b) is correct. Let's calculate the angular momentum using the equation from part (b):

$$L = I\omega = \frac{I}{\sqrt{mR^2 + I}}\sqrt{2mgh}.$$

This is indeed the largest for the largest I.

6 [10 points] You pull a 100N crate all the way up a frictionless 7.00m high 30.0 degree slope, using a frictionless pulley, as shown in the Figure.



Assuming the crate moves at a small constant speed while you pull it up, how much work (approximately) will you have done in the end?

(a) -1400J, (b) -700J, (c) -350J, (d) 0, (e) 350J, (f) 700J, (g) 1400J

Answer: Answer (f) is correct: either use the increase of potential energy $W = mgh =$ 700J to get that answer, or use $W = \int \vec{F} \cdot d\vec{x} = 100\text{N} \sin(30°) \times 7.0\text{m}/\sin(30°) = 700\text{J}$.

7 [20+7* points] A rock of mass m=70.0kg is dropped straight down from a height h =70.0m. Calculate the speed with which it hits the ground (ignoring friction) in three different ways:

(I) Within Newtonian physics, approximating the initial potential gravitational energy of the rock by mgh.

(II) Within Newtonian physics but modeling the Earth as a spherically symmetric mass distribution with radius R_E.

(III) Using Special Relativity and assuming mgh for the initial potential energy of the rock.

Extra credit*: Calculate the *relative difference* between answers (II) and (I), and that between answers (III) and (I).

Answers:

(I) $mv^2/2 = mgh$, which implies $v = \sqrt{2gh} = 37.1\text{m/s}$.

(II) $mv^2/2 = -GmM/(R_E + h) + GmM/R_E$, where $g = GM/R_E^2$ (so you don't have to know G and M_E here!). This gives the same answer (to 3 digits) $v = 37.1\text{m/s}$.

(III) $(\gamma - 1)mc^2 = mgh$ yields the same answer (to 3 digits): $v = 37.1\text{m/s}$.

One point of the question is that the answers will be virtually the same, since the approximation $U = mgh$ is very good as long as h is small compared to R_E, and since the Newtonian approximation for $K = mv^2/2$ should be very good as long as $v \ll c$.

Extra credit: For answer (II) we can approximate

$$\frac{GmM}{R_E} - \frac{GmM}{R_E + h} \approx \frac{GmM}{R_E}(1 - \frac{1}{1 + h/R_E}) \approx mgh(1 - \frac{1}{2}h/R_E)),$$

so that

$$v \approx \sqrt{2gh} \times \sqrt{1 - h/(2R_E)} \approx \sqrt{2gh} \times (1 - h/(4R_E)),$$

so that the relative correction to the answer to (I) is $-h/(4R_E) \approx -2.74 \times 10^{-6}$. For answer (III) we can approximate $(\gamma - 1)mc^2 \approx (1 + 3\beta^2/4)mv^2/2$, with $\beta = v/c$, so the relative difference between the answers to (III) and (I) is, similarly, approximately equal to $-3/8(v/c)^2 = -5.74 \times 10^{-15}$. Namely

$$v \approx \sqrt{2gh(1 - 3\beta^2/4)} \approx \sqrt{2gh}(1 - 3\beta^2/8).$$

8 [10+5* points] A space ship moving in the negative x-direction at a relativistic speed $v = 0.7c$ according to an inertial observer S, passes S at time $t_0 = 0$ in $x_0 = 0$ ("event 0"), and passes a small space station (at rest in S's frame) located at $x_1 = -0.7$ls (light seconds) at $t_1 = 1$s ("event 1"), all according to S. Explain why event 1 is later than event 0 in *any* other frame of reference S', even if S' is moving faster than the space ship according to S. Extra credit* for explaining the same fact in different ways.

Answer:

Here are four different ways to get the same answer:

(i) The interval between events 1 and 0: $c^2\Delta t^2 - \Delta\vec{r}^2 = (1^2 - (0.7)^2)\text{ls}^2 = 0.5\text{ls}^2$ is an invariant, meaning it's the same for all inertial observers. In particular, it is positive for all observers, so for all observers there is a nonzero time difference between the two events. This by itself doesn't prove yet that $\Delta t' > 0$, but by continuity [when going to a slightly different observer's speed], the time interval cannot change sign. (In fact, the *shortest* amount of time between the two events is $\tau = \sqrt{0.5}\text{s} \approx 0.7\text{s}$, which is the time between the events in the rest frame of the space ship.)

(ii) You could draw a space-time diagram: the line (plane) connecting events 0 and 1 is at an angle less than 45 degrees with the time ct axis. For any inertial observer the lines (planes) of equal time are at an angle more than 45 degrees with the ct axis, so event 1 is always *above any* equal-time line (plane) going through event 0.

(iii) You could use a causality argument: since events 1 and 0 *could* be causally connected, with event 0 being the cause of event 1, event 1 must be later than its cause, event 0, for any observer.

(iv) Finally, you could use the Lorentz transformations (the most straightforward way) to find

$$c\Delta t' = \gamma(c\Delta t - \beta\Delta x).$$

Since $|\beta| < 1$ and $c\Delta t > \Delta x$, we always have $\Delta t' > 0$.