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Tensor categories attached to cells in finite Weyl groups

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G – (semi)simple group
W – Weyl group of G
$S \subset W$ – simple reflections
$l : W \to \mathbb{Z}_{\geq 0}$ – length function

Hecke algebra H_W:
free module over $\mathbb{Z}[v, v^{-1}]$
basis T_x, $x \in W$
$T_x T_y = T_{xy}$ if $l(xy) = l(x) + l(y)$
(in particular $T_e = 1$)
$(T_s - v)(T_s + v^{-1}) = 0$ for $s \in S$

Kazhdan-Lusztig involution:
$h \mapsto \bar{h}$, automorphism of H_W
$\bar{v} = v^{-1}$, $\overline{T_s + v^{-1}} = T_s + v^{-1}$

Kazhdan-Lusztig basis:
1) $C_x \in T_x + \sum_{y \in W} v^{-1}\mathbb{Z}[v^{-1}]T_y$
2) $\overline{C_x} = C_x$.

Example: $C_e = T_e = 1$, $C_s = T_s + v^{-1}$

Structure constants:
$C_xC_y = \sum_{z\in W} h_{x,y,z}C_z$, $h_{x,y,z} \in \mathbb{Z}[v, v^{-1}]$

Lusztig's $a-$function: $a : W \to \mathbb{Z}_{\geq 0}$,
$h_{x,y,z} = \gamma_{x,y,z}v^{a(z)}$ + lower powers,
$\gamma_{x,y,z} \neq 0$ for some choice of $x, y \in W$.

Asymptotic Hecke ring J_W (Lusztig):
free $\mathbb{Z}-$module with basis t_x, $x \in W$
$t_xt_y = \sum_{z\in W} \gamma_{x,y,z}t_z$

Theorem (Lusztig, 1987):
1) $\gamma_{x,y,z} \in \mathbb{Z}_{\geq 0}$
2) J_W is associative
3) J_W has unit $1 = \sum_{d\in\mathcal{D}} t_d$

Question (Lusztig): Compute J_W explicitly (as a based ring)

Why we care: the ring J_W plays a significant role in the classification of
1) complex representations of $G(\mathbb{F}_q)$
2) character sheaves

$SL(2)-$**example:**
$C_e = 1$ and $C_s^2 = (v + v^{-1})C_s$
$h_{e,e,e} = 1$, $h_{e,s,s} = 1$, $h_{s,e,s} = 1$
$h_{s,s,s} = v + v^{-1}$
$a(e) = 0$, $a(s) = 1$

$t_e^2 = t_e$; $t_s^2 = t_s$; $t_e t_s = t_s t_e = 0$
in particular $1 = t_e + t_s$
$J_W = \mathbb{Z}t_e \oplus \mathbb{Z}t_s$ (direct sum of algebras)

For $A \subset W$ denote $J_A = \sum_{x \in A} \mathbb{Z}t_x \subset J_W$
Two sided cells:
finest partition $W = \sqcup C$ such that
$J_W = \bigoplus_C J_C$ is a direct sum of algebras

Example: A_1: $W = e \sqcup s$
A_2, B_2, G_2: $W = e \sqcup w_0 \sqcup$ (the rest)

Classification of two sided cells: known in all types (Lusztig, Barbasch-Vogan)

Refined Question: for a two sided cell C compute explicitly J_C (as a based ring)

Lusztig's finite group: $\Gamma = \Gamma(C)$
$\Gamma = \{e\}$ in type A
$\Gamma = (\mathbb{Z}/2\mathbb{Z})^k$ in type BCD
$\Gamma = S_k, 1 \leq k \leq 5$ in type GFE

Another based ring:
Γ – finite group, Y – finite $\Gamma-$set
$K_\Gamma(Y \times Y)$ – equivariant $K-$theory
convolution product: $p_{13*}(p_{12}^*(F_1) \otimes p_{23}^*(F_2))$
where $p_{ij} : Y \times Y \times Y \to Y \times Y$
basis: irreducible $\Gamma-$equivariant bundles on $Y \times Y$

Lusztig's Conjecture (1987):
For any two sided cell C there exists a finite $\Gamma(C)-$set $Y = Y(C)$ and an isomorphism of based rings:

$$J_C \simeq K_{\Gamma(C)}(Y \times Y)$$

Example (Lusztig, 1987):
$\Gamma(C) = \{e\}$ (e.g. type A)
$K(Y \times Y) = Mat_{|Y|}(\mathbb{Z})$
basis: matrix units
Conjecture holds true

Theorem (BFO):
Lusztig's conjecture holds true in general

Examples:

type B_2, unique cell with $\Gamma = \mathbb{Z}/2\mathbb{Z}$
$Y = \Gamma/\Gamma \sqcup \Gamma/\{e\}$
$|Y| = 3$, $|\Gamma \setminus Y| = 2$

type G_2, unique cell with $\Gamma = S_3$
$Y = S_3/S_3 \sqcup S_3/S_2$
$|Y| = 4$, $|\Gamma \setminus Y| = 2$

type F_4, unique cell with $\Gamma = S_4$
$Y = (S_4/S_4)^3 \sqcup (S_4/S_3)^3 \sqcup (S_4/S_2 \times S_2)^4 \sqcup S_4/S_2 \sqcup S_4/D_8$
$|Y| = 54$, $|\Gamma \setminus Y| = 12$

type E_8, unique cell with $\Gamma = S_5$
$Y = (S_5/S_5)^{420} \sqcup (S_5/S_4)^{756} \sqcup (S_5/D_8)^{168} \sqcup (S_5/S_2)^{70} \sqcup$
$\sqcup (S_5/S_3 \times S_2)^{1596} \sqcup (S_5/S_2 \times S_2)^{1092} \sqcup (S_5/S_3)^{378}$
$|Y| = 83160$, $|\Gamma \setminus Y| = 4480$

Categorification:
based ring = Grothendieck ring $K(\mathcal{C})$
of (additive) monoidal category $\mathcal{C}$
monoidal category:
category with tensor product, associativity
isomorphisms, unit object

Theorem (Lusztig, 1997): $J_C = K(\mathcal{C}_C)$

semisimple monoidal category $\mathcal{C}_C$ (over $\overline{\mathbb{Q}_l}$)
is defined via truncated convolution of
perverse sheaves on the flag variety of G

By definition:
$K_\Gamma(Y \times Y) = K(Coh_\Gamma(Y \times Y))$

$Coh_\Gamma(Y \times Y)$ (coherent $\Gamma-$equivariant sheaves
on $Y \times Y$) is semisimple monoidal category
(over any field of characteristic 0,
in particular $\overline{\mathbb{Q}_l}$)

Categorical Lusztig's Conjecture (1997):
There is a monoidal equivalence

$$\mathcal{C}_C \simeq Coh_{\Gamma(C)}(Y \times Y)$$

Theorem (BFO): Assume two sided cell C is not *exceptional*. Then categorical Lusztig's Conjecture holds true

Exceptional cells:
there is one (out of 35) in type E_7 and
two (out of 46) in type E_8
in all three cases $\Gamma(C) = \mathbb{Z}/2\mathbb{Z}$.

The equality $J_C = K_\Gamma(Y \times Y)$ for exceptional C was verified by Lusztig (1987)

$Y = \Gamma/\{e\} \sqcup \Gamma/\{e\} \sqcup \cdots$

More on exceptional cells.

Unit object of $\mathcal{C}_C$: $\mathbf{1} = \bigoplus_{d \in C \cap \mathcal{D}} \mathbf{1}_d$
$\mathbf{1}_d \otimes \mathbf{1}_d = \mathbf{1}_d; \quad \mathbf{1}_d \otimes \mathbf{1}_{d'} = 0$ if $d \neq d'$

Choose $\mathbf{1}_d$ (512 of them in smallest exceptional cell)

$\mathbf{1}_d \otimes \mathcal{C}_C \otimes \mathbf{1}_d$ is monoidal subcategory of $\mathcal{C}_C$

For an exceptional cell $\mathbf{1}_d \otimes \mathcal{C}_C \otimes \mathbf{1}_d$ has just two (isomorphism classes of) simple objects: $\mathbf{1}_d$ (unit object) and δ such that $\delta \otimes \delta \simeq \mathbf{1}_d$

Possible monoidal structures are classified by associativity isomorphism:
$\delta \otimes (\delta \otimes \delta) \to (\delta \otimes \delta) \otimes \delta$
equivalently, by a class in

$$H^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}$$

usual structure: same as in $Rep(\mathbb{Z}/2\mathbb{Z})$
unusual (twisted) structure: same as in $Rep(\widehat{sl}_2)_1$

Categorical Lusztig's Conjecture implies

$$\mathbf{1}_d \otimes \mathcal{C}_C \otimes \mathbf{1}_d \simeq Rep(\mathbb{Z}/2\mathbb{Z})$$

Theorem (O, in preparation):
For an exceptional two sided cell C we have $\mathbf{1}_d \otimes \mathcal{C}_C \otimes \mathbf{1}_d \simeq Rep(\widehat{sl}_2)_1$

Connection with finite $\mathcal{W}-$algebras

$e \in \mathfrak{g} = \mathrm{Lie}(G)$ nilpotent element

$\mathcal{W}_e$ finite $\mathcal{W}-$algebra (defined by Premet)

Recall that $Z(\mathcal{W}_e) = Z(\mathfrak{g}) = Z(U(\mathfrak{g}))$

Choose a **regular integral** central character $\chi : Z(\mathfrak{g}) \to \mathbb{C} = \mathrm{End}(\mathbb{C}_\chi)$

$\mathcal{W}_e^\chi := \mathcal{W}_e \otimes_{Z(\mathfrak{g})} \mathbb{C}_\chi$

Known facts:

(i) (Premet, Losev): $\mathcal{W}_e^\chi$ has nonzero finite dimensional module if and only if e is *special*

(ii) (Lusztig, Barbasch-Vogan): There is a natural bijection $e \mapsto C(e)$

$$\left\{ \begin{matrix} \text{special nilpotent} \\ G - \text{orbits in } \mathfrak{g} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{two sided} \\ \text{cells in } W \end{matrix} \right\}$$

Refinement of Conjecture by Premet:

Conjecture (Bezrukavnikov, O):

Assume e is special. There is a bijection:

$$\left\{ \begin{matrix} \text{simple finite dimensional} \\ \mathcal{W}_e^\chi - \text{modules} \end{matrix} \right\} \leftrightarrow Y(C(e))$$