MSRI–Eugene

Thursday Online Seminar

Incompressible symmetric tensor categories

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(jt with Dave Benson and Pavel Etingof)

Symmetric tensor categories

Representation categories

Base field: $k = \bar{k}$
Given a (finite) group G we can form
$\mathrm{Rep}(G) = \{$all finite dimensional representations of $G\}$

What kind of mathematical object is $\mathrm{Rep}(G)$?

- $\mathrm{Rep}(G)$ – abelian category
- $\mathrm{Rep}(G)$ has a bifunctor $\otimes$
- $\otimes$ is associative, commutative, unital $\leftarrow$ structures
- we have duality X^* $\leftarrow$ property of $\otimes$ and associativity
- we have forgetful tensor functor $\mathrm{Rep}(G) \to \mathrm{Vec}$

Thus $\mathrm{Rep}(G)$ is a rigid symmetric tensor category equipped with a (symmetric) tensor functor to Vec

Tannakian theory

Pre-Tannakian categories

$\mathcal{C}$: $k-$linear rigid symmetric $\otimes$ category satisfying

- $\mathcal{C}$ is abelian
- $\dim \mathrm{Hom}(X, Y) < \infty$
- length $(X) < \infty$
- $\mathbf{1}$ is simple

Definition

Fiber functor: exact symmetric $\otimes$ functor $\mathcal{C} \to \mathrm{Vec}$

$\mathcal{C}$ is Tannakian if it is pre-Tannakian and admits a fiber functor.

Theorem (Grothendieck, Saavedra Rivano, Deligne-Milne)

Assume $\mathcal{C}$ is Tannakian. Then $\mathcal{C} = Rep(G)$ for some (unique) affine group scheme G.

super Tannakian categories

Example

sVec – pre-Tannakian but not Tannakian
Reminder: sVec $=$ Rep$(\mathbb{Z}/2)$ but with modified commutativity constraint

Definition

super fiber functor: exact symmetric $\otimes$ functor $\mathcal{C} \to$ sVec
$\mathcal{C}$ is super Tannakian: pre-Tannakian and admits a super fiber functor.

Example

G affine super group scheme ($=$ super commutative Hopf algebra)
then Rep(G) is super Tannakian
More generally: $z \in G$, $z^2 = 1$, $Ad(z)$ =parity automorphism of G
Rep$(G, z) = \{$ objects X of Rep(G) such that the action of z is the parity automorphism of $X\}$
then Rep(G, z) is super Tannakian

super Tannakian theory and generalization

Theorem (Deligne)

Assume $\mathcal{C}$ is super Tannakian. Then $\mathcal{C} = Rep(G, z)$ for some affine super group scheme G and z as above.

Generalization

Let $F : \mathcal{C} \to \mathcal{D}$ be an exact symmetric $\otimes$ functor.
Then $\mathcal{C} = \mathrm{Rep}(G, \pi)$ where G is an affine group scheme in $\mathcal{D}$
π is the fundamental group of $\mathcal{D}$
In other words, $\mathcal{C}$ can be expressed in terms of "group theory in $\mathcal{D}$".

Question: What are categories which can't be expressed in terms of "group theory" in smaller categories?
Equivalently, which categories do not admit $\otimes$ functors to smaller categories?

Incompressible categories

Surjective functors

Exact $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is surjective if any object of $\mathcal{D}$ is a subquotient of $F(X)$
Any functor $F : \mathcal{C} \to \mathcal{D}$ factorizes $\mathcal{C} \to \mathrm{Im}(F) \to \mathcal{D}$ where $\mathcal{C} \to \mathrm{Im}(F)$ is surjective and $\mathrm{Im}(F) \to \mathcal{D}$ is an embedding.

Definition

A pre-Tannakian $\mathcal{C}$ is incompressible if any surjective $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence for any pre-Tannakian $\mathcal{D}$

Equivalently, $\mathcal{C}$ is incompressible if any exact $\otimes$ functor $F : \mathcal{C} \to \mathcal{D}$ is an embedding

Example

Vec   sVec

Any more examples?

Characteristic zero

Definition

We say that $\mathcal{C}$ is of sub-exponential growth if for any $X \in \mathcal{C}$ there is $a_X \in \mathbb{R}$ such that $\mathrm{length}(X^{\otimes n}) \leq a_X^n$.

Theorem (Deligne)

Assume char $k = 0$ and let $\mathcal{C}$ be pre-Tannakian of sub-exponential growth. Then $\mathcal{C}$ is super Tannakian. In particular, Vec and sVec are the only incompressible categories of sub-exponential growth.

Example

Deligne categories $\mathrm{Rep}(GL_t), \mathrm{Rep}(O_t), \mathrm{Rep}(S_t)$ $(t \in k)$ are categories of super-exponential growth.
They typically admit surjective functors like $\mathrm{Rep}(S_t) \to \mathrm{Rep}(S_{t-1})$.

Conjecture: No more incompressible categories in characteristic zero.

Semisimplification

Let $\mathcal{T}$ be a rigid symmetric monoidal category (perhaps non-abelian)

Negligible morphisms

We say that $f : X \to Y$ is negligible if for any $g : Y \to X$ we have $\mathrm{Tr}(fg) = 0$.
Negligible morphisms form a $\otimes$ ideal $\mathcal{N}$.
Define $\overline{\mathcal{T}}$: the same objects as in $\mathcal{T}$ but $\mathrm{Hom}_{\overline{\mathcal{T}}}(X, Y) = \mathrm{Hom}_{\mathcal{T}}(X, Y)/\mathcal{N}$.
$\overline{\mathcal{T}}$ is again rigid symmetric monoidal category

Theorem (U. Jannsen)

Assume $\dim Hom(X, Y) < \infty$ *and any nilpotent endomorphism in* $\mathcal{T}$ *has trace zero. Then* $\overline{\mathcal{T}}$ *is semisimple (and so abelian). Moreover*

$$\text{Irreducibles of } \overline{\mathcal{T}} \leftrightarrow \text{Indecomposables of } \mathcal{T} \text{ of nonzero dimension.}$$

Remark: assume $F : \mathcal{T} \to \mathcal{C}$ is a $\otimes$ functor to abelian $\mathcal{C}$. Then any nilpotent endomorphism in $\mathcal{T}$ has trace zero.

Semisimple Verlinde categories

Assume char $k = p > 0$.

S. Gelfand-Kazhdan and Georgiev-Mathieu

G – semisimple group, e.g. $G = SL_n$
Let $\mathcal{T} = \{$ tilting $G-$modules $\}$
Then $\mathrm{Ver}(G) := \overline{\mathcal{T}}$ is a nice $\otimes$ category; it has finitely many irreducibles provided $p \geq$ Coxeter number of G (e.g. $h(SL_n) = n$)

Example

$\mathrm{Ver}_p := \mathrm{Ver}(SL_2)$
Simple objects $L_1 = \mathbf{1}, L_2, \ldots, L_{p-1}$
$L_2 \otimes L_i = L_{i-1} \oplus L_{i+1}$ with convention $L_0 = L_p = 0$
This implies: $L_{p-1} \otimes L_{p-1} = \mathbf{1}$ and $\langle \mathbf{1}, L_{p-1} \rangle = \mathrm{sVec}$.
For $p = 5$: $L_3 \otimes L_3 = \mathbf{1} \oplus L_3$ Fibonacci category Fib
$\mathrm{Ver}_2 = \mathrm{Vec}$; $\mathrm{Ver}_3 = \mathrm{sVec}$; $\mathrm{Ver}_5 = \mathrm{Fib} \boxtimes \mathrm{sVec}$

Alternatively $\mathrm{Ver}_p =$ semisimplification of $\mathrm{Rep}(\mathbb{Z}/p)$

Fact: the category Ver_p is incompressible.

Theorem (V.O.)

For any pre-Tannakian $\mathcal{C}$ which is semisimple with finitely many irreducibles there exists a $\otimes$ functor $\mathcal{C} \to Ver_p$. In particular there is no other incompressible fusion categories.

Conjecture: there are no other semisimple incompressible categories (at least of sub-exponential growth).

What about non-semisimple examples?

Example

For $p = 2$ Venkatesh constructed an example $\mathcal{V}$ by modifying the commutativity constraint in $\mathrm{Rep}(\mathbb{Z}/2)$

Benson-Etingof

$p = 2$ sequence $\mathcal{C}_0 = \mathrm{Vec} \subset \mathcal{C}_1 = \mathcal{V} \subset \mathcal{C}_2 = \mathcal{T}(SL_2)/\mathcal{I}_2 \subset \mathcal{C}_3 \subset \dots$

Technology: Hopf algebras (in categories) and graded extensions

$p > 2$

More on tilting modules for SL_2

V – tautological 2-dimensional representation of SL_2
$\mathcal{T}(SL_2) :=$ additive $\otimes$ category generated by V
$\{$indecomposables of $\mathcal{T}(SL_2)\} = \{$indecomposable summands of $V^{\otimes n}\}$
Fact: indecomposables are classified by highest weight
Thus we have $T_0 = \mathbf{1}$, $T_1 = V$, T_2, $T_3, \ldots$
Steinberg modules: $T_0, T_{p-1}, T_{p^2-1}, \ldots, \mathrm{St}_r = T_{p^r-1}, \ldots$
Each St_r generates a (thick) tensor ideal $\mathcal{P}_r = \langle T_{p^r-1}, T_{p^r}, T_{p^r+1}, \ldots \rangle$
Each thick tensor ideal above gives tensor ideal $\mathcal{I}_r$: $\mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \ldots$
Fact: Any tensor ideal in $\mathcal{T}(SL_2)$ is one of $\mathcal{I}_r$

Quotients

Define $\mathcal{T}_{p,r} := \mathcal{T}(SL_2)/\mathcal{I}_r$, e.g. $\mathcal{T}_{p,1} = \mathrm{Ver}_p$
$\mathcal{T}_{p,r}$ is non-semisimple and non-abelian for $r > 1$ (except $r = 2$ and $p = 2$)
$\mathcal{T}_{p,r}$ contains tensor ideal $\mathcal{P}_{r-1}$ (and $\mathcal{I}_{r-1} = \mathcal{I}_{r-1}/\mathcal{I}_r$)

Main Theorem

Observation: (Benson-Etingof): their category $\mathcal{C}_{2r}$ contains $\mathcal{T}_{2,r}$;
$\{$ the ideal $\mathcal{P}_{r-1}\} = \{$ subcategory of projective objects in $\mathcal{C}_{2r}\}$

Theorem (Benson-Etingof-O., Coulembier)

There exists a unique pre-Tannakian category Ver_{p^n} containing $\mathcal{T}_{p,n}$ and such that $\mathcal{P}_{n-1}$ coincides with the ideal of projective objects. The category Ver_{p^n} is incompressible.

Split morphisms

A morphism (in additive category) $f : X \to Y$ is split if it is projection to a direct summand followed by an inclusion of a direct summand

Example

If X and Y are indecomposable, f is split $\Leftrightarrow$ f is an isomorphism or $f = 0$

Exercise. Let P be a projective object and $f : X \to Y$ be any morphism Then $\mathrm{id}_P \otimes f : P \otimes X \to P \otimes Y$ is split

Splitting ideals

Let $\mathcal{T}$ be a rigid ~~symmetric~~ monoidal category (perhaps non-abelian), $\dim \mathrm{Hom}(X, Y) < \infty$

Let $\mathcal{P} \subset \mathcal{T}$ be a thick tensor ideal

We say that $\mathcal{P}$ is splitting ideal if for any morphism $f : X \to Y$ in $\mathcal{T}$ and $P \in \mathcal{P}$ the morphism $\mathrm{id}_P \otimes f$ is split

General construction

Given splitting ideal $\mathcal{P} \subset \mathcal{T}$ as above we construct abelian rigid tensor category $\mathcal{C} \supset \mathcal{P}$ such that $\mathcal{P}$ is subcategory of projective objects

Hint on construction of $\mathcal{C}$: complexes of objects of $\mathcal{P}$

Some challenges: What is unit object of $\mathcal{C}$?

Why $\mathcal{C}$ is rigid?

Key Lemma

The ideal $\mathcal{P}_{n-1} \subset \mathcal{T}_{p,n}$ is splitting

Wanted: more examples of splitting tensor ideals!

Properties of Ver_{p^n}

Projectives and Cartan matrix

Projective objects: T_i with $p^{n-1} - 1 \le i < p^n - 1$; $\# = p^n - p^{n-1}$

We set $P_i = T_{i-1}$ where $i = [i_1 i_2 \dots i_n]_p$ has precisely n digits ($i_1 \neq 0$!)

Cartan matrix $C_{ij} := \dim \mathrm{Hom}(P_i, P_j)$

Negative digits game: make some digits negative

$[23045]_p \rightsquigarrow 2(-3)0(-4)5 = 2p^4 - 3p^3 - 4p + 5$

Descendants of i: all positive numbers you get in this way

E.g. $[23045]_p$ has $2^3 = 8$ descendants

Tubbenhauer-Wedrich: $C_{ij} = |\{\text{descendants of } i\} \cap \{\text{descendants of } j\}|$

Exercise: $C_{ij} = 0$ or power of 2

Exercise: $\det C_{ij}$ = power of p

Embeddings

We have $\mathrm{Ver}_p \subset \mathrm{Ver}_{p^2} \subset \mathrm{Ver}_{p^3} \subset \dots$

For $p > 2$, $\mathrm{Ver}_{p^n} = \mathrm{Ver}_{p^n}^+ \boxtimes \mathrm{sVec}$

More properties of Ver_{p^n}

Grothendieck ring ($p > 2$)

$K(\mathrm{Ver}^+_{p^n}) = \mathbb{Z}[\xi + \xi^{-1}]$ where ξ is a primitive p^n-th root of 1
$K(\mathrm{Ver}_{p^n}) = \mathbb{Z}[\xi + \xi^{-1}][\mathbb{Z}/2]$

Simples

Some simples: $T_0 = \mathbf{1}, T_1, \ldots, T_{p-1}$ are simples $T_i^{[n]}$ in Ver_{p^n} $(n > 1)$
Tensor Product Theorem: for $i = [i_1 \ldots i_n]$ with $i_1 \neq p - 1$
$L_i = T_{i_1}^{[1]} \otimes T_{i_2}^{[2]} \ldots T_{i_n}^{[n]}$ is simple (and this is a complete list)
$P(L_i) = P_s$ where $s = [(i_1 + 1)i_2^* \ldots i_n^*]$ where $i^* = p - 1 - i$
Extensions ($p > 2$): $\mathrm{Ext}^1(L_i, L_j) = 0$ or k
$\mathrm{Ext}^1(L_i, L_j) \neq 0 \Leftrightarrow i$ and j differ only in two consecutive digits, of which the first ones differ by 1 and the second ones add up to $p - 2$
Blocks: $n(p - 1)$ of them of sizes $1, p - 1, p^2 - p, \ldots, p^{n-1} - p^{n-2}$
Ver_{p^n} is a mod p reduction of semisimple category in characteristic zero
Corollary: $C = DD^T$ where D is the decomposition matrix

Some open questions

Module categories

Question 1. What are exact module categories over Ver_{p^n}?

Extensions

Question 2. What is $\mathrm{Ext}^\bullet(\mathbf{1},\mathbf{1})$?

More examples?

Question 3. Are there any other incompressible categories?

Universality

Let $\mathrm{Ver}_{p^\infty} = \cup_n \mathrm{Ver}_{p^n}$

Let $\mathcal{C}$ be a pre-Tannakian category of sub-exponential growth

Question 4. Is there an exact tensor functor $\mathcal{C} \to \mathrm{Ver}_{p^\infty}$?

Thanks for listening!