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Hopf Algebras and Tensor Categories

The group of modular extensions of a symmetric tensor category

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Braided fusion categories

Fusion categories

$= \mathbb{C}-$linear (or $k-$linear) semisimple rigid tensor categories, $\dim \mathrm{Hom}(X, Y) < \infty$, finitely many simple objects, $\mathbf{1}$ is simple
Measure of size: *Frobenius-Perron dimension* $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{R}_{>0}$

Double braiding

$\mathcal{C}$ – braided fusion category
Braiding $c_{XY} : X \otimes Y \to Y \otimes X$
$\mathcal{C}' := \{X \in \mathcal{C} | c_{XY} \circ c_{YX} = \mathrm{Id} \ \ \forall Y \in \mathcal{C}\}$ ← Müger center

Two extremes

$\mathcal{C} = \mathcal{E}$ is symmetric if $\mathcal{E}' = \mathcal{E}$
$\mathcal{C} = \mathcal{M}$ is *non-degenerate* or modular if $\mathcal{M}' = \mathrm{Vec} = \langle \mathbf{1} \rangle$

Remark: $\mathcal{C}'$ is always symmetric

(Minimal) Modular Extensions

$\mathcal{C}$ – braided fusion category
Question: Can we embed $\mathcal{C}$ into modular $\mathcal{M}$?
Answer: Yes, e.g. $\mathcal{C} \subset \mathcal{Z}(\mathcal{C})$ $\leftarrow$ Drinfeld center
Question: What are most economical embeddings $\mathcal{C} \subset \mathcal{M}$?

Theorem (Müger, Drinfeld-Gelaki-Nikshych-O)

$\mathrm{FPdim}(\mathcal{M}) \geq \mathrm{FPdim}(\mathcal{C})\mathrm{FPdim}(\mathcal{C}')$.

Definition

Minimal modular extension: $\mathcal{C} \subset \mathcal{M}$ with
$\mathrm{FPdim}(\mathcal{M}) = \mathrm{FPdim}(\mathcal{C})\mathrm{FPdim}(\mathcal{C}')$.

Example

$\mathrm{FPdim}(\mathcal{Z}(\mathcal{C})) = \mathrm{FPdim}(\mathcal{C})^2$
Hence $\mathcal{C} \subset \mathcal{Z}(\mathcal{C})$ is minimal if and only if $\mathcal{C}$ is symmetric

$\mathcal{M}ext(\mathcal{C})$ and $\mathcal{M}ext(\mathcal{E})$

Notation: $\mathcal{M}ext(\mathcal{C})$ – set of (minimal) modular extensions of $\mathcal{C}$
Remark: $\mathcal{M}ext(\mathcal{C})$ might be empty! (examples by Drinfeld)

Theorem (Lan-Kong-Wen)

(i) For symmetric $\mathcal{E}$, $\mathcal{M}ext(\mathcal{E})$ is an abelan group
(ii) $\mathcal{M}ext(\mathcal{C})$ is a torsor over $\mathcal{M}ext(\mathcal{E})$ where $\mathcal{E} = \mathcal{C}'$

Example

(i) G – finite group and $\mathcal{E} = \mathrm{Rep}_{\mathbb{C}}(G)$
Then $\mathcal{M}ext(\mathcal{E}) = H^3(G, \mathbb{C}^\times)$
Explicitly: for $\omega \in H^3(G, \mathbb{C}^\times)$, $\mathrm{Rep}(G) \subset \mathcal{Z}(\mathrm{Vec}_G^\omega)$
(ii) $\mathcal{E} = \mathrm{sVec}$ then $\mathcal{M}ext(\mathcal{E}) = \mathbb{Z}/16\mathbb{Z}$ (Kitaev)
16-fold way (Bruillard, Galindo, Hagge, Ng, Plavnik, Rowell, Wang):
if $\mathcal{C}' = \mathrm{sVec}$ then $|\mathcal{M}ext(\mathcal{C})| = 16$ or (open question!) 0

Classification of symmetric tensor categories

More symmetric categories

Modify braiding in Rep(G): pick a central involution $z \in G$ and set

$$c'_{XY}(x \otimes y) = (-1)^{mn} y \otimes x$$

$$\text{if } zx = (-1)^m x, zy = (-1)^n y$$

Notation: Rep(G, z)

Example

Rep(G, e) = Rep(G)
Super vector spaces: sVec = Rep($\mathbb{Z}/2\mathbb{Z}, z$) ($z$ nontrivial)

Theorem (Grothendieck, Saavedra Rivano, Doplicher–Roberts, Deligne)

A symmetric fusion category over $\mathbb{C}$ is of the form Rep(G, z).

$\mathcal{M}ext(\mathrm{Rep}(G,z))$

Category **zGrp**:
Objects: (G,z) where $z \in G$ is central of order ≤ 2
Morphisms from (G,z) to (G',z'):
homomorphisms $\phi : G \to G'$ such that $\phi(z) = z'$

Theorem (R. Usher)

$\mathcal{M}ext(Rep(G,z))$ is a contravariant functor **zGrp** $\to \mathbb{Z}-$**Mod**

Question: What is this functor?

Example

Assume z is nontrivial.
Obvious morphism: $(\mathbb{Z}/2\mathbb{Z}, z) \to (G,z)$
Hence homomorphism:
$\mathcal{M}ext(\mathrm{Rep}(G,z)) \to \mathcal{M}ext(\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z},z)) = \mathbb{Z}/16\mathbb{Z}$

Homomorphism: $\mathcal{M}ext(\mathrm{Rep}(G,z)) \to \mathbb{Z}/16\mathbb{Z}$

Theorem (R. Usher)

The homomorphism $\mathcal{M}ext(Rep(G,z)) \to \mathbb{Z}/16\mathbb{Z}$ is surjective iff

$$1 \to \langle z \rangle = \mathbb{Z}/2\mathbb{Z} \to G \to G/\langle z \rangle \to 1$$

splits.

Question: How to characterize central extensions giving other images?

Example

$G = \mathbb{Z}/4\mathbb{Z}$, $z \in G$ is nontrivial
Then $|\mathcal{M}ext(\mathrm{Rep}(G,z))| = 32$
Image of $\mathcal{M}ext(\mathrm{Rep}(G,z)) \to \mathbb{Z}/16\mathbb{Z}$ is of order 8
Which group this is?

More context

Back to $\mathcal{E} = \text{Rep}(G)$

Recall that $\mathcal{M}ext(\text{Rep}(G)) = H^3(G, \mathbb{C}^\times)$
Also $H^4(G, \mathbb{C}^\times)$ – group of obstructions for $\mathcal{M}ext(\mathcal{C})$ being non-empty (where $\mathcal{C}' = \text{Rep}(G)$)
$H^1(G, \mathbb{C}^\times) =$ group of invertible objects in $\text{Rep}(G)$
$H^2(G, \mathbb{C}^\times) =$ group of invertible module categories over $\text{Rep}(G)$

Definition

For symmetric fusion category $\mathcal{E}$ we set
$H^1_{sym}(\mathcal{E}) =$ group of invertible objects in $\mathcal{E}$ (?)
$H^2_{sym}(\mathcal{E}) =$ group of invertible module categories over $\mathcal{E}$
$H^3_{sym}(\mathcal{E}) = \mathcal{M}ext(\mathcal{E})$
$H^4_{sym}(\mathcal{E})$ – conjectural group of obstructions for $\mathcal{M}ext(\mathcal{C})$ being non-empty (where $\mathcal{C}' = \mathcal{E}$)

Question: In what sense $H^i_{sym}(\mathcal{E})$ is a cohomology theory?

Non-semisimple generalization

$\mathcal{E}$ – finite symmetric tensor category
$\mathcal{M}ext(\mathcal{E})$ – group of minimal *factorizable* extensions $\mathcal{E} \subset \mathcal{M}$
Fact: if $\mathcal{E}$ is fusion over $\mathbb{C}$ we get the same notion as before
Fact: in characteristic $p > 0$ new definition of $\mathcal{M}ext(\mathcal{E})$ is forced:
$\mathcal{Z}(\mathcal{E})$ might be non-semisimple!

Example

$\mathcal{E}$ over $\mathbb{C}$: representations of additive supergroup $(V_{odd}, +) =$
representations of super Hopf algebra $\wedge^\bullet(V)$
Gelaki-Sebbag: $\mathcal{M}ext(\mathcal{E}) = \mathbb{Z}/16\mathbb{Z}$

Another interesting example: $\mathcal{E} = \mathrm{Ver}_p$ (Verlinde category) open!

Thanks for listening!