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Multi-fusion categories of Harish-Chandra bimodules

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Plan of the talk

1. Harish-Chandra (bi)modules.
2. Associated varieties and tensor product modulo "smaller size".
3. Tensor categories and multi-fusion categories.
4. Actions of Harish-Chandra bimodules (Whittaker modules and finite $W-$algebras).
5. Sheaves.

Harish-Chandra modules

$G_{\mathbb{R}}$ – real semi-simple Lie group, e.g. $SL(n, \mathbb{R})$
Harish-Chandra (1953): many questions about continuous complex representations of $G_{\mathbb{R}}$ can be reduced to pure algebra.

$\mathfrak{g} = \mathrm{Lie}(G_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$, $U(\mathfrak{g})$ – universal enveloping algebra
$K \subset G_{\mathbb{R}}$ – maximal compact subgroup, e.g. $SO(n, \mathbb{R}) \subset SL(n, \mathbb{R})$

Definition

A $(\mathfrak{g}, K)-$module (or Harish-Chandra module) is a space V with actions of $\mathfrak{g}$ and K such that
1. V is algebraic $K-$module, i.e. V is a union of finite dimensional $K-$modules.
2. The actions are compatible: $\mathfrak{g}$-action is $K-$equivariant and the differential of $K-$action agrees with $Lie(K) \subset \mathfrak{g}-$action.
3. V is finitely generated $U(\mathfrak{g})-$module.

Complex groups and bimodules

$G_{\mathbb{C}}$ – complex simply connected semi-simple Lie group, e.g. $SL(n,\mathbb{C})$
Let us consider $G_{\mathbb{C}}$ as a real Lie group

$\mathfrak{g} = \mathrm{Lie}(G_{\mathbb{C}})$; $\mathrm{Lie}(G_{\mathbb{C}}) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus \mathfrak{g}$
representation of $\mathrm{Lie}(G_{\mathbb{C}}) \otimes_{\mathbb{R}} \mathbb{C} \Leftrightarrow$ module over $U(\mathfrak{g} \oplus \mathfrak{g}) = U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$
$x \mapsto -x$ induces $U(\mathfrak{g}) \simeq U(\mathfrak{g})^{op}$, so $U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g}) \simeq U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})^{op}$
Thus $\mathrm{Lie}(G_{\mathbb{C}}) \otimes_{\mathbb{R}} \mathbb{C}-$representation is the same as $U(\mathfrak{g})-$bimodule

We can choose $K \subset G_{\mathbb{C}}$ such that $\mathrm{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C} \subset \mathrm{Lie}(G_{\mathbb{C}}) \otimes_{\mathbb{R}} \mathbb{C}$ is the diagonal $\Delta\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$, e.g. $K = SU(n) \subset SL(n,\mathbb{C})$
M – $U(\mathfrak{g})-$bimodule; adjoint action: $ad(x)m := xm - mx$
$U(\mathfrak{g})-$bimodule is algebraic if it is a union of finite dimensional $\mathfrak{g}$-modules with respect to the adjoint action.

Example

$U(\mathfrak{g})$ is algebraic (use PBW filtration) and $U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$ is not.

Harish-Chandra bimodules

Definition

A Harish-Chandra bimodule over $\mathfrak{g}$ is a finitely generated $U(\mathfrak{g})-$bimodule which is algebraic.

Lemma

If M and N are Harish-Chandra bimodules then so is $M \otimes_{U(\mathfrak{g})} N$.

- The tensor product $\otimes_{U(\mathfrak{g})}$ is associative
- $U(\mathfrak{g})$ is the unit for this tensor product

Thus the category $\mathcal{H}$ of Harish-Chandra bimodules is a tensor category.

Remark. If M is a Harish-Chandra bimodule over $\mathfrak{g}$ and N is $(\mathfrak{g}, K)-$module then $M \otimes_{U(\mathfrak{g})} N$ is also $(\mathfrak{g}, K)-$module
Thus the category $\mathcal{H}$ acts on the category of $(\mathfrak{g}, K)-$modules.

Central characters and simple Harish-Chandra bimodules

$Z(\mathfrak{g}) \subset U(\mathfrak{g})$ center of the universal enveloping algebra
$Z(\mathfrak{g})$ acts on an irreducible $\mathfrak{g}-$module via central character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$
${}_{\chi_1}\mathcal{H}_{\chi_2} \subset \mathcal{H}$ – full subcategory where the left $Z(\mathfrak{g})-$action factors through χ_1 and the right $Z(\mathfrak{g})-$action factors through χ_2
Any irreducible Harish-Chandra bimodule is contained in a unique ${}_{\chi_1}\mathcal{H}_{\chi_2} \subset \mathcal{H}$

${}_{\chi_1}\mathcal{H}_{\chi_2} \otimes_{U(\mathfrak{g})} {}_{\chi_3}\mathcal{H}_{\chi_4} \subset {}_{\chi_1}\mathcal{H}_{\chi_4}$ and ${}_{\chi_1}\mathcal{H}_{\chi_2} \otimes_{U(\mathfrak{g})} {}_{\chi_3}\mathcal{H}_{\chi_4} = 0$ unless $\chi_2 = \chi_3$
$\mathcal{H}(\chi) := {}_{\chi}\mathcal{H}_{\chi}$ is tensor subcategory of $\mathcal{H}$
unit object: $U(\mathfrak{g})_\chi := U(\mathfrak{g})/\mathrm{Ker}(\chi)U(\mathfrak{g})$

Convention: χ is integral regular, e.g. $\chi = \chi_0$ trivial central character

Theorem (Bernstein-S. Gelfand, Enright, Joseph)

Irreducible bimodules in $\mathcal{H}(\chi) \leftrightarrow$ elements of the Weyl group W.

Proof uses Bernstein-Gelfand-Gelfand category $\mathcal{O}$.

Associated varieties

$M \in \mathcal{H}$, $M_0 \subset M$ finite dimensional subspace which generates M and which is invariant under the adjoint action
$U(\mathfrak{g})_0 \subset U(\mathfrak{g})_1 \subset \cdots \subset U(\mathfrak{g})$ PBW filtration
$M_n = U(\mathfrak{g})_n M_0 \Rightarrow$ filtration $M_0 \subset M_1 \subset \cdots \subset M$

Associated graded

$\mathrm{gr}M$ is a finitely generated module over $\mathrm{gr}U(\mathfrak{g}) = S^\bullet(\mathfrak{g})$
Moreover, this module is equivariant with respect to $G_{\mathbb{C}}-$action

Let us identify $\mathfrak{g}^* = \mathrm{Spec}(S^\bullet(\mathfrak{g}))$ with $\mathfrak{g}$ via the Killing form

Definition

The associated variety $V(M)$ is the support of $\mathrm{gr}M$ in $\mathfrak{g}$.

- $V(M) = V(L) \cup V(K)$ for a s.e.s. $0 \to L \to M \to K \to 0$
- $V(M \otimes_{U(\mathfrak{g})} N) \subset V(M) \cap V(N)$

Filtration by nilpotent orbits

Nilpotent orbits

$x \in \mathfrak{g}$ is nilpotent if $ad(x) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent
Example. $x \in sl(n, \mathbb{C})$ is nilpotent $\Leftrightarrow x^n = 0$
$\mathcal{N} \subset \mathfrak{g}$ is the nilpotent cone, i.e. the set of all nilpotent elements
Dynkin+Kostant: $\mathcal{N}$ consists of finitely many $G_{\mathbb{C}}-$orbits
Example. nilpotent orbits in $sl(n, \mathbb{C}) \leftrightarrow$ partitions of n
For $\mathbb{O} \subset \mathcal{N}$, $\bar{\mathbb{O}}$ is its closure; partial order: $\mathbb{O}' \leq \mathbb{O} \Leftrightarrow \mathbb{O}' \subset \bar{\mathbb{O}}$

- for $M \in {}_{\chi_1}\mathcal{H}_{\chi_2}$ we have $V(M) \subset \mathcal{N}$. Moreover,

Theorem (Borho-Brylinsky, Joseph)

For irreducible $M \in \mathcal{H}$, $V(M)$ is irreducible, i.e. $V(M) = \bar{\mathbb{O}}$.

$\mathcal{H}(\chi)_{\leq \mathbb{O}}$ – full subcategory of $\mathcal{H}(\chi)$ consisting of M with $V(M) \subset \bar{\mathbb{O}}$
$\mathcal{H}(\chi)_{< \mathbb{O}}$ – full subcategory of $\mathcal{H}(\chi)_{\leq \mathbb{O}}$ consisting of M with $V(M) \neq \bar{\mathbb{O}}$
Both $\mathcal{H}(\chi)_{\leq \mathbb{O}}$ and $\mathcal{H}(\chi)_{< \mathbb{O}}$ are Serre subcategories
$\mathcal{H}(\chi)_{\leq \mathbb{O}}$ is closed under $\otimes_{U(\mathfrak{g})}$; $\mathcal{H}(\chi)_{< \mathbb{O}}$ is "ideal" with respect to $\otimes_{U(\mathfrak{g})}$

Cell categories

Serre quotients

We can form $\tilde{\mathcal{H}}(\chi)_{\mathbb{O}} = \mathcal{H}(\chi)_{\leq \mathbb{O}}/\mathcal{H}(\chi)_{<\mathbb{O}}$
Tensor products $\otimes_{U(\mathfrak{g})}$ descends to $\otimes : \tilde{\mathcal{H}}(\chi)_{\mathbb{O}} \times \tilde{\mathcal{H}}(\chi)_{\mathbb{O}} \to \tilde{\mathcal{H}}(\chi)_{\mathbb{O}}$

- it is not clear whether $\tilde{\mathcal{H}}(\chi)_{\mathbb{O}}$ has a unit object

$\mathcal{H}(\chi)_{\mathbb{O}}$ – full subcategory of $\tilde{\mathcal{H}}(\chi)_{\mathbb{O}}$ consisting of semisimple objects

Theorem (Joseph, Bezrukavnikov-Finkelberg-O, Losev)

$\mathcal{H}(\chi)_{\mathbb{O}}$ *is closed under* $\otimes$.

$\mathcal{H}(\chi)_{\mathbb{O}}$ has a unit object: let $Pr(\chi)_{\mathbb{O}}$ be the (finite) set of primitive ideals in $U(\mathfrak{g})_\chi$ with $V(U(\mathfrak{g})/I) = \bar{\mathbb{O}}$; then $\mathbf{1} = \oplus_{I \in Pr(\chi)_{\mathbb{O}}} U(\mathfrak{g})/I$

Theorem (Bezrukavnikov-Finkelberg-O, Losev-O)

$\mathcal{H}(\chi)_{\mathbb{O}}$ *is a multi-fusion category.*

We will call $\mathcal{H}(\chi)_{\mathbb{O}}$ cell category associated with $\mathbb{O}$

Definition (MacLane)

Tensor category: quadruple $(\mathcal{C}, \otimes, a, \mathbf{1})$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, $a_{X,Y,Z} : (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ is an associativity constraint, $\mathbf{1}$ is the unit object.

1. Pentagon axiom: the following diagram commutes for all $W, X, Y, Z \in \mathcal{C}$:

$$
\begin{array}{ccccc}
 & & ((W \otimes X) \otimes Y) \otimes Z & & \\
 & \swarrow^{a_{W,X,Y} \otimes \mathrm{id}_Z} & & \searrow^{a_{W \otimes X, Y, Z}} & \\
(W \otimes (X \otimes Y)) \otimes Z & & & & (W \otimes X) \otimes (Y \otimes Z) \\
\downarrow{\scriptstyle a_{W, X \otimes Y, Z}} & & & & \downarrow{\scriptstyle a_{W,X,Y \otimes Z}} \\
W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow{\mathrm{id}_W \otimes a_{X,Y,Z}} & & W \otimes (X \otimes (Y \otimes Z))
\end{array}
$$

2. Unit axiom: both functors $\mathbf{1} \otimes ?$ and $? \otimes \mathbf{1}$ are isomorphic to the identity functor.

Rigidity

For $X \in \mathcal{C}$ its right dual is $X^* \in \mathcal{C}$ together with $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$ such that the compositions equal the identities:

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$

$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X,X^*,X}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^*$$

Definition

$\mathcal{C}$ is rigid if any $X \in \mathcal{C}$ has right and left duals.

Example (s)

1. $\mathcal{C} = \mathrm{Bimod}(R)$ bimodules over a ring R: tensor product is $\otimes_R$, $\mathbf{1} = R$.
$M \in \mathcal{C}$ has right dual $\Leftrightarrow$ M is f.g. projective as left $R-$module.
2. $\mathcal{C} = \mathrm{End}(\mathcal{A})$ functors from a category $\mathcal{A}$ to itself; tensor product is composition, $\mathbf{1} = \mathrm{Id}$. $F \in \mathcal{C}$ has a dual $\Leftrightarrow$ adjoint of F exists.
3. $\mathcal{C} = \mathrm{Mod}(R)$ modules over a commutative ring R; e.g. vector spaces over a field.
$M \in \mathcal{C}$ has right dual $\Leftrightarrow$ M is f.g. projective $\Leftrightarrow$ M has left dual.

Example (continued)

4. (H. Sinh) Objects: elements of a group A; $\mathrm{Hom}(g,h)=\emptyset$ if $g\neq h$, $\mathrm{Hom}(g,g)=S$ where S is an abelian group. $g\otimes h=gh$, $\alpha\otimes\beta=\alpha\beta$ for $g,h\in A$, $\alpha,\beta\in S$. Associativity constraint: $\omega_{g,h,k}\in S$ for any $g,h,k\in A$. Pentagon axiom $\Leftrightarrow \partial\omega=1$, i.e. ω is a 3-cocycle on A with values in S. Tensor structures are parameterized by $H^3(A,S)$.

5. R – algebra over k with trivial center. Consider the category of invertible bimodules over R (morphisms are *isomorphisms* of bimodules). This category is tensor equivalent to category from (4). $A=\mathrm{Pic}(R)$ group of isomorphism classes of invertible bimodules (= non-commutattive Picard group of R); $S=k^\times$. Associator $\omega\in H^3(\mathrm{Pic}(R),k^\times)$.

5a. $\mathrm{Pic}(R)\supset\mathrm{Out}(R)$: $M_\phi=R$, $(a,b)\cdot c=ac\phi(b)$.
Let $1\neq\phi\in\mathbb{Z}/2\mathbb{Z}\subset\mathrm{Out}(R)$, so $\phi^2=Ad(g)$.
Exercise. (i) $\phi(g)=\pm g$; (ii) $\omega|_{\mathbb{Z}/2\mathbb{Z}}\neq 0\Leftrightarrow\phi(g)=-g$; (iii) Let $\phi(g)=-g$. Then $M^\phi\not\simeq M$ for any $M\in\mathrm{Irr}(R)$.

$R=\mathbb{C}\langle g,x,y\rangle/(xy-yx-1,g^2-1,gx+xg,gy+yg)$,
$\phi(g)=-g,\phi(x)=-y,\phi(y)=x$.

Multi-fusion categories

Definition (Etingof, Nikshych, O)

Tensor category $\mathcal{C}$ over k is multi-fusion if it is rigid and semi-simple with finitely many simple objects. $\mathcal{C}$ is fusion if in addition $\mathbf{1}$ is simple.

Example (char(k)=0)

0. Vec – finite dimensional vector spaces.
1. Rep(A) – f.d. representations of finite group A.
2. Vec_A – f.d. $A-$graded vector spaces. Thus simple objects are $k_a, a \in A$ and $k_a \otimes k_b = k_{ab}$. Generalization: Vec_A^ω – same as Vec_A but $\omega \in H^3(A, k^\times)$ is used as the associator.
3. Bimod(R) where R is semisimple, e.g. $R = k \oplus k$. $\mathbf{1} = R$ is not simple.
4. Y is a finite set with $A-$action. $\mathrm{Coh}_A(Y \times Y)$ – $A-$equivariant vector bundles (or coherent sheaves) on $Y \times Y$. Convolution product: $F_1 * F_2 = p_{13*}(p_{12}^*(F_1) \otimes p_{23}^*(F_2))$ where $p_{ij} : Y \times Y \times Y \to Y \times Y$.

Exercise. What is the number of simple summands in $\mathbf{1} \in \mathrm{Coh}_A(Y \times Y)$?

Module categories

The categories $\mathrm{Coh}_A(Y \times Y)$ are not closed under the operation of taking full tensor subcategory. For example Vec_B^ω with $\omega \neq 0$ is usually not of the form $\mathrm{Coh}_A(Y \times Y)$ but it can be found as a subcategory in a suitable $\mathrm{Coh}_A(Y \times Y)$.

Definition

Let $\mathcal{C}$ be a tensor category and $\mathcal{M}$ be a category. We say that $\mathcal{M}$ is a module category over $\mathcal{C}$ (or that $\mathcal{C}$ *acts* on $\mathcal{M}$) if we have a tensor functor $\mathcal{C} \to \mathrm{End}(\mathcal{M})$. Equivalently, we have a bifunctor $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ with associativity constraint satisfying suitable axioms.

Example

Consider $\mathcal{C} = \mathrm{Vec}_A^\omega$. Let $B \subset A$ and $\psi \in Z^2(B, k^\times)$ be such that $\partial\psi = \omega|_B$. Then $R_B = \oplus_{b \in B} k_b$ acquires a structure of associative algebra in $\mathcal{C}$. Then $\mathcal{M}(B, \psi) = \{$ right R_B-modules in $\mathcal{C}\}$ is naturally a module category over $\mathcal{C}$. Simple objects of $\mathcal{M}(B, \psi) \leftrightarrow A/B$.

Dual categories

Convention: If $\mathcal{C}$ is a multi-fusion category then any module category is assumed to be semisimple with finitely many simple objects.

Definition

Let $\mathcal{M}$ be a module category over $\mathcal{C}$. Then $\mathcal{C}^*_{\mathcal{M}} := \mathrm{End}_{\mathcal{C}}(\mathcal{M})$ is called dual category of $\mathcal{C}$ with respect to $\mathcal{M}$.

Properties (Müger+Etingof, Nikshych, O)

- $\mathcal{C}^*_{\mathcal{M}}$ is multi-fusion category
- $\mathcal{C}^*_{\mathcal{M}}$ is fusion $\Leftrightarrow$ $\mathcal{M}$ is indecomposable module category over $\mathcal{C}$
- $(\mathcal{C}^*_{\mathcal{M}})^*_{\mathcal{M}} \simeq \mathcal{C}$
- $\mathcal{C} \sim \mathcal{C}^*_{\mathcal{M}}$ is an equivalence relation (2-Morita equivalence)
- $\mathcal{C} \xrightarrow{F} \mathcal{D}$ tensor functor and $\mathcal{M}$ is module category over $\mathcal{D}$. Then we have $\mathcal{D}^*_{\mathcal{M}} \xrightarrow{F^*} \mathcal{C}^*_{\mathcal{M}}$
- let us say that F is injective if it is fully faithful and surjective if any object of $\mathcal{D}$ is a subquotient of $F(X)$. F injective $\Leftrightarrow$ F^* surjective

Convolution with twists

Example

$\mathcal{C} = \mathrm{Vec}_A$ and $\mathcal{M} = \oplus_i \mathcal{M}(B_i, 1)$.
Then $\mathcal{C}^*_{\mathcal{M}} = \mathrm{Coh}_A(Y \times Y)$ where $Y = \sqcup_i A/B_i$ (so $\mathcal{M} = \mathrm{Coh}(Y)$).

Generalization

Let $\mathcal{C} = \mathrm{Vec}_A^\omega$ and $\mathcal{M} = \oplus_i \mathcal{M}(B_i, \psi_i)$. We consider $\mathcal{C}^*_{\mathcal{M}}$ as cohomologically twisted version of $\mathrm{Coh}_A(Y \times Y)$.
Notation: $\mathcal{C}^*_{\mathcal{M}} = \mathrm{Coh}_{A,\omega}(Y \times Y)$. Note that the information about ψ_i's is implicitly contained in Y; Y is cohomologically twisted $A-$set.

Lemma

Let $\mathcal{C} \subset Coh_A(Y \times Y)$ be a full multi-fusion subcategory such that $\mathcal{M} = Coh(Y)$ is indecomposable over $\mathcal{C}$. Then there exists a surjective functor $F : Vec_A \to Vec_{\bar{A}}^\omega$ such that the action of Vec_A on $\mathcal{M}$ factors through F and such that $\mathcal{C} = Coh_{\bar{A},\omega}(Y \times Y) = (Vec_{\bar{A}}^\omega)^*_{\mathcal{M}} \subset (Vec_A)^*_{\mathcal{M}}$.

Whittaker modules

Let $e \in \mathfrak{g}$ be a nilpotent element.
Jacobson-Morozov: $\exists h, f \in \mathfrak{g}$ s.t. $[h,e] = 2e, [h,f] = -2f, [e,f] = h$.
$\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}(n), \;\; \mathfrak{g}(n) = \{x \in \mathfrak{g} | [h,x] = nx\}$.
E.g. $e \in \mathfrak{g}(2)$ and $f \in \mathfrak{g}(-2)$.
$x, y \mapsto (e, [x,y])$ non-degenerate skew-symmetric bilinear form on $\mathfrak{g}(-1)$.
Pick a lagrangian subspace $\ell \subset \mathfrak{g}(-1)$ and set $\mathfrak{m} = \mathfrak{m}_\ell = \ell \oplus \bigoplus_{i\le -2} \mathfrak{g}(i)$.
Then $\xi(x) = (x, e)$ is a Lie algebra homomorphism $\mathfrak{m} \to \mathbb{C}$.
$\mathfrak{m}_\xi :=$ Lie subalgebra of $U(\mathfrak{g})$ spanned by $x - \xi(x), x \in \mathfrak{m}$.

Definition (Moeglin)

We say that $\mathfrak{g}-$module is Whittaker if the action of $\mathfrak{m}_\xi$ on it is locally nilpotent. Wh – full subcategory of Whittaker $\mathfrak{g}-$modules.

$\widetilde{\mathrm{Skr}}$: Wh $\to$ Vect, $M \mapsto \{v \in M | \mathfrak{m}_\xi v = 0\}$.
$U(\mathfrak{g}, e) = \mathrm{End}(\widetilde{\mathrm{Skr}})$ – Premet's finite $W-$algebra.
Skryabin (also Gan, Ginzburg and Losev): Skr : Wh $\xrightarrow{\sim}$ Mod$(U(\mathfrak{g}, e))$
Remark: $U(\mathfrak{g}, e)$ does not depend on choice of $\ell \subset \mathfrak{g}(-1)$.

Action of the cell categories on Whittaker modules

Lemma

For $M \in \mathcal{H}$ and $N \in Wh$, $M \otimes_{U(\mathfrak{g})} N \in Wh$. Thus $\mathcal{H}$ acts on Wh.

• Let ${}^{\chi}\mathrm{Wh}$ be the full subcategory of $M \in \mathrm{Wh}$ such that $Z(\mathfrak{g})-$action factors through a central character χ. Then $\mathcal{H}(\chi)$ acts on ${}^{\chi}\mathrm{Wh}$.
${}^{\chi}\mathrm{Wh}^f$ – full subcategory of ${}^{\chi}\mathrm{Wh}$ consisting of semisimple M such that $\mathrm{Skr}(M)$ is finite dimensional ($\simeq$ semisimple f.d. $U(\mathfrak{g}, e)-$modules).

Theorem (Losev)

Let $\mathbb{O} = G_{\mathbb{C}}e$. For $M \in \mathcal{H}(\chi)_{\leq \mathbb{O}}$ and $N \in {}^{\chi}Wh^f$, $M \otimes_{U(\mathfrak{g})} N \in {}^{\chi}Wh^f$. For $M \in \mathcal{H}(\chi)_{< \mathbb{O}}$ and $N \in {}^{\chi}Wh^f$, $M \otimes_{U(\mathfrak{g})} N = 0$. Thus the cell category $\mathcal{H}(\chi)_{\mathbb{O}}$ acts on ${}^{\chi}Wh^f$.

Let $Q = Z_{G_{\mathbb{C}}}(e, f, h)$. Then Q acts on $U(\mathfrak{g}, e)$ and on ${}^{\chi}\mathrm{Wh}^f$.
$Q-$action on ${}^{\chi}\mathrm{Wh}^f$ commutes with $\mathcal{H}(\chi)_{\mathbb{O}}-$action.
$Q^0 \subset Q$ the unit component. The action of Q^0 on ${}^{\chi}\mathrm{Wh}^f$ is trivial.
Warning: this does not imply that $C(e) := Q/Q^0$ acts on ${}^{\chi}\mathrm{Wh}^f$.

Irreducible finite dimensional $U(\mathfrak{g}, e)$-modules

Let us choose a finite subgroup $A \subset Q$ which surjects to Q/Q^0. Then ${}^\chi\mathrm{Wh}^f$ is a module category over Vec_A.

Theorem (Losev,O)

The functor $\mathcal{H}(\chi)_{\mathbb{O}} \to (\mathrm{Vec}_A)^*_{{}^\chi\mathrm{Wh}^f} = \mathrm{End}_{\mathrm{Vec}_A}({}^\chi\mathrm{Wh}^f)$ *is fully faithful.*

${}^\chi\mathrm{Wh}^f$ as module category over Vec_A

Y – set of isomorphism classes of irreducible f.d. $U(\mathfrak{g}, e)-$modules.
A acts on Y; moreover we have data of cohomologically twisted $A-$set.
Thus $(\mathrm{Vec}_A)^*_{{}^\chi\mathrm{Wh}^f} = \mathrm{Coh}_A(Y \times Y)$ and $\mathcal{H}(\chi)_{\mathbb{O}} \subset \mathrm{Coh}_A(Y \times Y)$

Corollary

There is a quotient $\bar{A}$ *of* A *and* $\omega \in H^3(\bar{A}, \mathbb{C}^\times)$ *such that the action of* Vec_A *on* ${}^\chi\mathrm{Wh}^f$ *factors through tensor functor* $\mathrm{Vec}_A \to \mathrm{Vec}^\omega_{\bar{A}}$ *and the action on* ${}^\chi\mathrm{Wh}^f$ *induces tensor equivalence* $\mathcal{H}(\chi)_{\mathbb{O}} \simeq \mathrm{Coh}_{\bar{A},\omega}(Y \times Y)$.

Complements

- $\mathcal{H}(\chi)_{\mathbb{O}} \neq 0 \Leftrightarrow$ the nilpotent orbit $\mathbb{O}$ is special in the sense of Lusztig.
- The quotient map $A \to \bar{A}$ factorizes through $A \subset Q \to Q/Q^0 = C(e)$. $\bar{A}$ is Lusztig's quotient of $C(e)$ (defined for any special nilpotent orbit).
- Irr. summands of $\mathbf{1} \in \mathcal{H}(\chi)_{\mathbb{O}} \leftrightarrow$ primitive ideals I with $V(U(\mathfrak{g})/I) = \bar{\mathbb{O}}$. Irr. summands of $\mathbf{1} \in \mathrm{Coh}_{\bar{A},\omega}(Y \times Y) \leftrightarrow \bar{A}-$orbits ($=Q-$orbits) in Y. Hence irreducible f.d. $U(\mathfrak{g}, e)_{\chi}-$modules which give rise to the same primitive ideal are $Q-$conjugated (Losev).
- Recall that irreducible objects of $\mathcal{H}(\chi) \leftrightarrow W$. It follows from Joseph's irreducibility theorem that $\mathrm{Irr}(\mathcal{H}(\chi)) = \sqcup_{\mathbb{O}} \mathrm{Irr}(\mathcal{H}(\chi)_{\mathbb{O}})$. Hence we have a partition of W indexed by special nilpotent orbits. This is known to coincide with partition into Kazhdan-Lusztig two sided cells. Each two sided cell is in turn partitioned into left cells and into right cells. This corresponds to partitions $\mathrm{Irr}(\mathcal{C}) = \sqcup_i \mathbf{1}_i \otimes \mathrm{Irr}(\mathcal{C}) = \sqcup_i \mathrm{Irr}(\mathcal{C}) \otimes \mathbf{1}_i$ where $\mathbf{1} = \oplus_i \mathbf{1}_i$ which holds for any multi-fusion category $\mathcal{C}$.
- $Y = \sqcup_i \bar{A}/B_i$ where $B_i \subset \bar{A}$ is well-defined up to conjugacy. These are Lusztig's subgroups attached to any left cell.

• $\oplus_{\mathbb{O}} K(\mathcal{H}(\chi)_{\mathbb{O}}) =: J$ is known to be asymptotic Hecke algebra (Lusztig).
Lusztig's isomorphism: $J \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}[W]$. Thus any $\mathbb{Q}-$module over $K(\mathcal{H}(\chi)_{\mathbb{O}})$ gives rise to a $W-$module. For example $K(\mathcal{H}(\chi)_{\mathbb{O}} \otimes \mathbf{1}_i) \otimes \mathbb{Q}$ is constructible representation attached to a left cell.
Also $K(\mathrm{Coh}(Y))$ is a module over $K(\mathrm{Coh}_{\bar{A},\omega}(Y \times Y))$
Dodd: there is $W \times C(e)-$equivariant embedding of $K(\mathrm{Coh}(Y)) \otimes \mathbb{Q}$ into Springer representation $H^{top}(\mathcal{B}_e)$.

• The 3-cocycle $\omega \in H^3(\bar{A}, \mathbb{C}^\times)$ is almost always zero. $\omega \neq 0$ iff the corresponding two sided cell is exceptional. This happens only in types E_7 and E_8; in this case $\bar{A} = \mathbb{Z}/2\mathbb{Z}$. Proof requires theory of character sheaves.

• Assume that $\omega = 0$. Then ${}^{\chi}\mathrm{Wh}^f = \oplus_i \mathcal{M}(B_i, \psi_i)$. It can be shown that the cocycles ψ_i are all trivial.

• There is a conjectural description (Losev,O) of what happens in the case of χ which is no longer integral. The calculations suggest that in this case nontrivial 2-cocycles show up often.

• Further results: Losev gave formulas for dimensions of irreducible modules in ${}^{\chi}\mathrm{Wh}^f$ and proved that they coincide with Goldie ranks of quotients by primitive ideas.

Derived convolution

F – algebraically closed field (possibly of positive characteristic)
X – algebraic variety over F
Sheaves on X form a category over field k:
(a) $D-$modules: char(F)=0, $k = F$
(b) perverse constructible sheaves in classical topology: $F = \mathbb{C}$, any k
(c) perverse constructible $\ell-$adic sheaves: $\ell \neq 0$ in F, $k = \bar{\mathbb{Q}}_\ell$

G – semisimple group over F of the same Dynkin type as $\mathfrak{g}$
$\mathcal{B}$ – flag variety of G ($\mathcal{B} = G/B$ where B is a Borel subgroup)
Simple $G-$equivariant sheaves on $\mathcal{B} \times \mathcal{B} \leftrightarrow G-$orbits on $\mathcal{B} \times \mathcal{B} \overset{Bruhat}{\longleftrightarrow} W$;
$w \leftrightarrow I_w$
Convolution $*$: $F_1 * F_2 = p_{13*}(p_{12}^*(F_1) \otimes p_{23}^*(F_2))$ (use derived categories!)
Decomposition Theorem (Beilinson, Bernstein, Deligne and Gabber) $\Rightarrow$
$I_u * I_v \simeq \bigoplus_{w,i} I_w[i]^{n_{u,v}^w(i)}$
$C_u C_v = \sum_{w,i} n_{u,v}^w(i) t^i C_w$ – Hecke algebra (over $\mathbb{Z}[t, t^{-1}]$) with
Kazhdan-Lusztig basis

Asymptotic Hecke algebra and truncated convolution

$a(w) = max\{i | n_{u,v}^w(i) \neq 0$ for some $u, v\}$ – Lusztig's $a-$function
$t_u t_v = \sum_w n_{u,v}^w(a(w)) t_w$ – Lusztig's asymptotic Hecke algebra J (over $\mathbb{Z}$)
Lusztig: J is associative with unit; $J \otimes \mathbb{Q} \simeq \mathbb{Q}[W]$
$J = \oplus_C J_C$ – sum over two sided cells in W; $a|_C = const =: a(C)$

Multi-fusion category $\mathcal{J}_C$: simple objects $I_w, w \in C$
truncated convolution: $I_u \bullet I_v := \oplus_{w \in C} I_w^{n_{u,v}^w(a(C))}$

Beilinson-Bernstein: $D-$modules on $\mathcal{B} \simeq \mathfrak{g}-$modules with central character χ_0.
Corollary: $G-$equivariant $D-$modules on $\mathcal{B} \times \mathcal{B} \simeq \mathcal{H}(\chi_0)$.
Beilinson-Ginzburg: we can change equivalence above and make it tensor
Corollary (Bezrukavnikov, Finkelberg, O): $D-$module version of $\mathcal{J}_C \simeq \mathcal{H}_{\mathbb{O}}$.
Theorem (Bezrukavnikov, Finkelberg, O): $\mathcal{J}_C \simeq \mathrm{Coh}_{\bar{A},\omega}(Y \times Y)$ for any F.

Character sheaves and Drinfeld center

$G-$equivariant sheaves on $\mathcal{B}\times\mathcal{B} = B\times B-$equivariant sheaves on G
Such sheaves are $\Delta(B)-$equivariant $= Ad(B)-$equivariant
$\Gamma_B^G : Ad(B)-$equivariant sheaves $\to Ad(G)-$equivariant sheaves
Simple constituents of $\Gamma_B^G(I_w) =:$ (unipotent) character sheaves (Lusztig)

$\mathcal{C}$ – tensor category $\Rightarrow$ Drinfeld center $\mathcal{Z}(\mathcal{C})$:
Objects of $\mathcal{Z}(\mathcal{C})$ = pairs (X,ϕ) where $\phi : X\otimes ? \simeq ?\otimes X$
Müger, O: $\mathcal{Z}(\mathcal{C}^*_{\mathcal{M}}) \simeq \mathcal{Z}(\mathcal{C})$ for a multi-fusion category $\mathcal{C}$
Example: $\mathcal{Z}(\mathrm{Coh}_{\bar{A},\omega}(Y\times Y)) \simeq \mathcal{Z}(\mathrm{Vec}^{\omega}_{\bar{A}})$ – (twisted) Drinfeld double

Observation: the functor Γ_B^G is formally similar to functor $I : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$
Bezrukavnikov, Finkelberg, O: using $D-$modules (so $char(F) = 0$)
Ben-Zvi, Nadler: in the setting of infinity categories
Lusztig: using mixed sheaves (for any F)
Corollary: unipotent character sheaves $\leftrightarrow \sqcup_{\mathbb{O}}\mathrm{Irr}(\mathcal{Z}(\mathrm{Vec}_{\bar{A}}))$.

Coxeter groups

W – finite crystallographic Coxeter group
What about more general Coxeter groups?

W – affine Weyl group
Lusztig: two sided cells in $W \leftrightarrow$ nilpotent orbit in $\mathfrak{g}$
Bezrukavnikov, O: $\mathcal{J}_C \simeq \mathrm{Coh}_Q(Y \times Y)$ (recall $Q = Z_{G_{\mathbb{C}}}(e, f, h)$)
Bezrukavnikov, Mirković: interpretation of the set Y in terms of unrestricted representations of $\mathfrak{g}$ in positive characteristic

W – infinite crystallographic group
Lusztig: category $\mathcal{J}_C$ makes sense; however
- infinite number of simple objects
- $\mathbf{1}$ might be "infinite direct sum"

Soergel+Elias, Williamson+Lusztig: $\mathcal{J}_C$ makes sense for any W!
- rigidity is not known; usually $\mathcal{J}_C$ is not a convolution category

Thanks for listening!