EXTENDED ROTATION ALGEBRAS: ADJOINING SPECTRAL PROJECTIONS TO ROTATION ALGEBRAS

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ABSTRACT. Denote by A_θ the rotation algebra corresponding to the rotation $2\pi\theta$. The C*-algebra $\mathcal{B}_\theta$ generated by A_θ together with certain spectral projections of the canonical unitary generators is studied. The C*-algebra $\mathcal{B}_\theta$ is shown to have a unique tracial state and to be nuclear provided that θ is irrational. Moreover, we study the ideal structure of the C*-algebra $\mathcal{B}_\theta$. In particular, it is shown that $\mathcal{B}_\theta$ is simple if neither the commutative sub-C*-algebra generated by the spectral projections of u in the question (assumed to be a set invariant under Adv) nor the corresponding commutative sub-C*-algebra associated to v contains non-zero minimal projections. In the second part of the paper, we study the extended rotation algebra $\mathcal{B}_\theta$ generated by the spectral projections (one for each unitary) corresponding to the half-open interval from 0 to θ. (The spectral projections for each half-open interval from $n\theta$ to $(n+1)\theta$ are then included for each integer n.) Using the simplicity of $\mathcal{B}_\theta$ for θ irrational, the natural field of C*-algebras on the unit circle with fibres $\mathcal{B}_\theta$ is shown to be continuous at irrational points. This field is lower semicontinuous on the whole circle. Much more useful is an upper semicontinuous field which is obtained by desingularizing this field at rational points on the circle. The fibres of the desingularized field at rational points are certain (computable) type I C*-algebras. Using this new field, we are able to show that $\mathcal{B}_\theta$ is an AF-algebra with $\mathrm{K}_0(\mathcal{B}_\theta) \cong \mathbb{Z} + \theta\mathbb{Z}$ for generic θ, in the sense of Baire Category, with the class of the unit being $1 \in \mathbb{Z}$.

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1. Introduction

A classical result of Pimsner and Voiculescu ([PV80]) is that the irrational rotation C*-algebra can be embedded in an AF-algebra with the same ordered K_0-group (and with the embedding giving rise to an isomorphism at the level of K_0). While a somewhat different proof of this can be obtained from the classification theorem for simple A$\mathbb{T}$-algebras ([Ell97]—it is in fact enough to use the real rank zero case given in [Ell93]), together with the fact that the irrational rotation C*-algebra is itself an A$\mathbb{T}$-algebra ([EE93]—in [EE93] it is also shown that the real rank is zero), in either approach the construction is far from canonical; infinitely many, necessarily arbitrary, choices must be made.

The purpose of the present article is to propose a canonical construction, and to show that this construction gives the desired AF-embedding at least in the case of a generic irrational rotation, in the sense of Baire Category.

This article is organized as follows. In Section 2, the extended rotation algebra is introduced, as the universal C*-algebra generated by the rotation algebra together with a specified set of spectral projections for each of the two canonical unitary generators (Definition 2.1). We investigate the universal property of the irrational extended rotation algebras. In particular, if there is no non-zero minimal projections, each irrational extended rotation algebras can be described as a certain universal algebra generated by the canonical unitaries and their logarithms (Corollary 2.10). In Section 3, by means of an averaging process, the extended rotation algebra is shown to have a unique tracial state, and the faithfulness of this state is shown to be equivalent to the simplicity of the extended rotation algebra. In Section 4, we make several remarks concerning the question of the nuclearity of the extended rotation algebra, and in Section 7, we use these remarks to show that any extended rotation algebra is nuclear.

In Section 5, we study the ideal structure of extended rotation algebras. It is show that the maximal ideal of an extended rotation algebra is always isomorphic to a direct sum of the algebra of compact operators, and moreover, it is exactly generated by the minimal projections of the commutative sub-C*-algebra generated by spectral projections of u and the commutative sub-C*-algebra generated by spectral projections of v. In particular, if the spectral projections are corresponding to half-open intervals with same orientation, the extended rotation algebra is simple. In Section 6, we make a remark on the universality of simple extended rotation algebras.

From Section 8, we consider the case of the extended rotation algebra $\mathcal{B}_\theta$ generated by the two spectral projections—one for each of the canonical unitary generators—corresponding to the half-open interval from 0 to θ. We construct a field of C*-algebras over the unit circle, which is upper semicontinuous—and continuous at each irrational point—, with fibre the extended rotation algebra at each irrational point, and with fibre a certain type I C*-algebra at each rational point. Using this upper semicontinuous field of C*-algebras, we show that there is a dense G_δ subset U of the circle such that $\mathcal{B}_\theta$ is the AF-algebra with

$$(\mathrm{K}_0(\mathcal{B}_\theta), \mathrm{K}_0^+(\mathcal{B}_\theta), [\mathbf{1}_{\mathcal{B}_\theta}]) \cong (\mathbb{Z}+\theta\mathbb{Z}, (\mathbb{Z}+\theta\mathbb{Z})\cap\mathbb{R}^+, 1)$$

for any $\theta \in U$ (Theorem 8.39).

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2. Preliminaries

Let A be a C*-algebra. By an ideal of A, we shall mean a closed two-sided ideal of A. Recall that if the only ideals of A are A itself and $\{0\}$, the C*-algebra A is said to be simple. If $\mathbf{1} \in A$, it is a simple fact that an ideal I of A has non-zero intersection with $\mathbb{C}\mathbf{1}$ if and only if $I = A$.

Let θ be a real number. Recall that the rotation algebra A_θ is the universal C*-algebra generated by two unitaries u and v satisfying the Heisenberg relation

$$uv = e^{2\pi i\theta} vu.$$

If $\theta = p/q$ with $(p, q) = 1$, A_θ is a C*-algebra of type I with spectrum $\mathbb{T}^2$, and the dimension of any irreducible representation of A_θ is q. If θ is irrational, the C*-algebra A_θ is simple, and has a unique tracial state. Moreover, in the case that θ is irrational, it was shown in [EE93] that A_θ is the inductive limit of a sequence of circle algebras (finite direct sums of matrix algebras over the circle), and furthermore (as a consequence), has real rank zero.

Fix an irrational number θ in $[0, 1]$. Let $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{j\in\Lambda_2}$ be two collections of (closed, open, or half open) subintervals of the unit circle $\mathbb{T}$, where Λ_1, Λ_2 are countable index sets (finite or infinite). Note that the spectrum of u and v is the unit circle. Given a faithful representation π of A_θ on a Hilbert space $\mathscr{H}$, let us still denote by f_i and g_j the spectral projections of $\pi(u)$ and $\pi(v)$ in $\mathcal{B}(\mathscr{H})$ corresponding to the specific collections of subintervals of $\mathbb{T}$. Let us refer to the C*-algebra

$$B_{\pi,\theta} := \mathrm{C}^*(\{\pi(u), \pi(v), f_i, g_j;\ i \in \Lambda_1, j \in \Lambda_2\}) \subset \mathcal{B}(\mathscr{H})$$

as the concrete extended rotation algebra obtained by adjoining these projections, specifically, with respect to the collections of intervals $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{j\in\Lambda_2}$ in $\mathbb{T}$ and the representation π of A_θ.

The C*-algebra $B_{\pi,\theta}$ in general depends on the representation π. However, we introduce the universal extended rotation algebra as follows.

Consider the C*-algebra $\mathrm{C}(\mathbb{T})$ as the canonical sub-C*-algebra of $\ell^\infty(\mathbb{T})$, and denote by σ the automorphism of $\ell^\infty(\mathbb{T})$ induced by

$$f(z) \mapsto f(e^{2\pi i\theta} z).$$

Note that $\mathrm{C}(\mathbb{T})$ is invariant under the action of σ. Still denote by f_i and g_j the spectral projections of the canonical unitary $f(z) = z$ in $\ell^\infty(\mathbb{T})$ corresponding to the intervals f_i and g_j of $\mathbb{T}$.

Consider the two commutative C*-algebras

$$\mathrm{C}(\Omega_u) := \mathrm{C}^*(\mathrm{C}(\mathbb{T}) \cup \{\sigma^{-k}(f_i);\ i \in \Lambda_1, k \in \mathbb{Z}\}) \subset \ell^\infty(\mathbb{T})$$

and

$$\mathrm{C}(\Omega_v) := \mathrm{C}^*(\mathrm{C}(\mathbb{T}) \cup \{\sigma^{k}(g_j);\ j \in \Lambda_2, k \in \mathbb{Z}\}) \subset \ell^\infty(\mathbb{T})$$

where Ω_u and Ω_v denote the spectra of these algebras. Denote by u and v the canonical generators of $\mathrm{C}(\mathbb{T})$ in $\mathrm{C}(\Omega_u)$ and $\mathrm{C}(\Omega_v)$ respectively.

Definition 2.1. For an irrational number θ, and two collections of subintervals $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{j\in\Lambda_2}$ of the unit circle $\mathbb{T}$, we shall refer to the universal C*-algebra generated by $\mathrm{C}(\Omega_u)$ and $\mathrm{C}(\Omega_v)$ with respect to the relations

(1) $uv = e^{2\pi i\theta}vu$,
(2) $u\sigma^k(g_j)u^* = \sigma^{k+1}(g_j)$ for any $j \in \Lambda_2$ and $k \in \mathbb{Z}$, and
(3) $v\sigma^{-k}(f_i)v^* = \sigma^{-k-1}(f_i)$ for any $i \in \Lambda_1$ and $k \in \mathbb{Z}$

as the (irrational) extended rotation algebra, and denoted by $\mathcal{B}_\theta$ $(= \mathcal{B}_\theta(\{f_i\}, \{g_j\}))$.

Concerning the question of rational extended rotation algebras, we shall define these in Section 8.1 as certain concrete C*-algebras.

It follows from the universality of $\mathcal{B}_\theta$ that the C*-algebra $B_{\pi,\theta}$ is a quotient C*-algebra of $\mathcal{B}_\theta$ for any representation π of A_θ.

In the rest part of this section, we shall investigate more details on generators and relations for the extended rotation algebras.

Definition 2.2. Given real numbers $a_1, a_2, ...$ will be said to be θ-independent if $a_i - a_j = n\theta$ mod $\mathbb{Z}$ for some $n \in \mathbb{Z}$ implies $i = j$, i.e., if the orbits of the points $e^{2\pi i a_1}, e^{2\pi i a_2}, ...$ under the rotation σ are distinct.

Consider the commutative C*-algebra $\mathrm{C}(\Omega_u)$. We have the following lemma.

Lemma 2.3. *There is a set of θ-independent real numbers $\{a_k;\ k \in \Lambda_u\}$ for some countable index set Λ_u (finite or infinite) such that the C*-algebra $\mathrm{C}(\Omega_u)$ is generated by*

$$\{\sigma^n(p_k), \sigma^n(e_k);\ n \in \mathbb{Z}, k \in \Lambda_u\},$$

where p_k is the spectral projection corresponding to $[a_k, a_k+\theta)$ or $(a_k, a_k+\theta]$, and e_k is the minimal projection corresponding to $\{a_k\}$ or zero. Moreover, there exists a non-zero minimal projection e_k in this decomposition if and only if there exists $l \in \Lambda_1$ such that the interval corresponding to f_l is closed with length $n\theta$ or $n(1-\theta)$ for some natural number n.

Proof. Since θ is irrational, the C*-algebra $\mathrm{C}(\Omega_u)$ is generated by

$$\{\sigma^n(f_l);\ n \in \mathbb{Z}, l \in \Lambda_1\},$$

where f_l is a spectral projection corresponding to a subinterval (half-open or closed) of $\mathbb{T}$. Denote by x_l and y_l the left endpoint and right endpoint of f_l respectively. Set $\mathcal{O}$ the orbit of $\{x_l, y_l;\ l \in \Lambda_1\}$. We then have

$$\mathcal{O} = \dot{\bigcup_{k\in\Lambda_u}} \mathcal{O}_k$$

with each O_k an orbit of a single point.

Pick a representative a_k from each $\mathcal{O}_k$. It is clear that $\{a_k;\ k \in \Lambda_u\}$ is θ-independent (their orbits are disjoint). Assume that the spectral projection f_l corresponds to a closed subinterval $[x_l, y_l]$ with $x_l \in \mathcal{O}_{k_1}$ and $y_l \in \mathcal{O}_{k_2}$. If $k_1 \neq k_2$, then there exist integers n_1 and n_2 such that if denote by g_1 and g_2 the spectral projections corresponding to $[x_l, x_l + n_1\theta)$ and $(y_l + n_2\theta, y_l]$ respectively, then $f_l = g_1 g_2$, in other words, f_l is generated by spectral projections corresponding to $[x_l, x_l + n_1\theta)$ and $(y_l + n_2\theta, y_l]$. Moreover, for any integer m, the spectral projection corresponding to $[x_l, x_l + m\theta)$ is generated by f_l. (There is an integer s such that $f_l + \sigma^s(f_l) - \sigma^s(f_l)f_l$ is a spectral projection corresponding to $[x_l, x_l + m'\theta)$ for some integer m', and hence $[x_l, x_l + m\theta)$ is generated by f_l.)

If $k_1 = k_2$, then the projection f_l can be generated by spectral projections corresponding to $[x_l, x_l + n_1\theta)$ and $\{\theta\}$, which is a minimal projection. Note that this minimal projection is also generated by f_l.

Repeating the argument above for each $f_l, l \in \Lambda_1$, one has that the spectral projection corresponding to f_l is generated by spectral projections $\{p_k;\ k \in \Lambda_u\}$, projections $\{e_k;\ k \in \Lambda_u\}$, together with their rotations, where p_k is the spectral projection corresponding to $[a_k, a_k + \theta)$ or $(a_k, a_k + \theta]$, and e_k is the minimal projection corresponding to $\{a_k\}$ or zero. The second part of the statement follows from the construction. □

Remark 2.4. We shall refer the points $\{a_k;\ k \in \Lambda_u\}$ above the *cutting points* of the canonical unitary u.

Moreover, in the case there is no non-zero minimal projections, the commutative C*-algebra $\mathrm{C}(\Omega_u)$ and $\mathrm{C}(\Omega_v)$ is independent on the choice of the orientation of half-open intervals.

Lemma 2.5. *Let $\{a_k;\ k \in \Lambda_u\}$ be a set of θ-independent real numbers. For each $k \in \Lambda_u$, let p_k be a spectral projection corresponding to a half-open interval with endpoints in the orbit of a_k under the rotation. Then the C*-algebra*

$$\mathrm{C}(\Omega_u) = C^*\{\sigma^n(p_k);\ k \in \Lambda_u, n \in \mathbb{Z}\}$$

is independent with the choice of the orientation of the half-open interval. More precisely, the map $p_k \mapsto p'_k$ by changing the orientation of the half-open interval induces the isomorphism.

Proof. Let us assume that p_1 is the spectral projection corresponding to $[a_1, a_1 + \theta)$. Set $p'_k = (a_1, a_1+\theta]$ and $p'_k = p_k$. In order to prove the theorem, it is enough to show that the map $p_1 \mapsto p'_1$, $p_k \mapsto p_k,\ k \neq 1$ induces an isomorphism from $\mathrm{C}^*\{\sigma^n(p_k);\ k \in \Lambda_u, n \in \mathbb{Z}\}$ to $\mathrm{C}^*\{\sigma^n(p'_k);\ k \in \Lambda_u, n \in \mathbb{Z}\}$, and it is enough to show that for any polynomial $P(x_k^n;\ n \in \mathbb{Z}, k \in \Lambda_u)$,

$$\|P(\sigma^n(p_k);\ n \in \mathbb{Z}, k \in \Lambda_u)\| = \|P(\sigma^n(p'_k);\ n \in \mathbb{Z}, k \in \Lambda_u)\|.$$

For polynomial P, one has

$$P(\sigma^n(p_k);\ n \in \mathbb{Z}, k \in \Lambda_u) = \sum_i (c_i \prod_{k \in \Lambda_u} e_{i,k}) = \sum_i (c_i e_{i,1} \prod_{\substack{k \in \Lambda_u \\ k \neq 1}} e_{i,k}) = \sum_i c_i e_{i,1} f_i,$$

where $e_{i,1}$ is a spectral projection corresponding to $[a_1 + m_i\theta, a_1 + n_i\theta)$ for some integers m_i and n_i, $e_{i,k}$ is a spectral projection corresponding to some half-open interval with endpoints in the orbits of a_k for $k \neq 1$, and f_i is a spectral projection corresponding to some interval with endpoints in the orbits of $\{a_k;\ k \neq 1\}$. Applying the polynomial P to p'_k, one then has

$$P(\sigma^n(p'_k);\ n \in \mathbb{Z}, k \in \Lambda_u) = \sum_i c_i e'_{i,1} f_i,$$

where $e'_{i,1}$ is a spectral projection corresponding to $(a_1 + m_i\theta, a_1 + n_i\theta]$ for some integers m_i and n_i.

Since $a_k;\ k \in \Lambda_u$ are θ-independent, there exists a point $t \in \mathbb{T}$ which is not in the orbit of a_1 such that

$$\|P(\sigma^n(p_k);\ n \in \mathbb{Z}, k \in \Lambda_u)\| = |P(\sigma^n(p_k);\ n \in \mathbb{Z}, k \in \Lambda_u)(t)| = \left|\sum_i c_i e_{i,1}(t) f_i(t)\right|.$$

Since t is not in the orbit of a_1, one has

$$\left|\sum_i c_i e_{i,1}(t) f_i(t)\right| = \left|\sum_i c_i e'_{i,1}(t) f_i(t)\right| \leq \|P(\sigma^n(p'_k);\ n \in \mathbb{Z}, k \in \Lambda_u)\|,$$

and therefore

$$\|P(\sigma^n(p_k);\ n \in \mathbb{Z}, k \in \Lambda_u)\| \leq \|P(\sigma^n(p'_k);\ n \in \mathbb{Z}, k \in \Lambda_u)\|.$$

The same argument shows that

$$\|P(\sigma^n(p'_k);\ n \in \mathbb{Z}, k \in \Lambda_u)\| \leq \|P(\sigma^n(p_k);\ n \in \mathbb{Z}, k \in \Lambda_u)\|,$$

and hence

$$\|P(\sigma^n(p_k);\ n \in \mathbb{Z}, k \in \Lambda_u)\| = \|P(\sigma^n(p'_k);\ n \in \mathbb{Z}, k \in \Lambda_u)\|,$$

as desired. □

In Definition 2.1, relations among the projections $\{f_i\}$ (or $\{g_j\}$) are hidden in the (concrete) commutative C*-algebra $\mathrm{C}(\Omega_u)$ (or $\mathrm{C}(\Omega_v)$). However, using Lemma 2.3, one can change the generators $\{f_i\}$ and $\{g_j\}$ to the projections corresponding to intervals with endpoints in a single orbit. Then, in the case there are no non-zero minimal projections (in the commutative C*-algebras $\mathrm{C}(\Omega_u)$ and $\mathrm{C}(\Omega_v)$), the extended rotation algebra can be characterized entirely by generators and relations in the following way.

First, let $\{a_k;\ k \in \Lambda_u\}$ be a set of θ-independent numbers, where Λ_u is a countable index set (finite of infinite). Consider the set of generators

$$\mathcal{G}' = \{u, d_{k,n};\ n \in \mathbb{Z}, k \in \Lambda_u\}$$

and the set of relations $\mathcal{R}'$ consisting of

(1) u is a unitary, and each d_k is a projection,
(2) $ud_{k,n} = d_{k,n}u$ for any $n \in \mathbb{Z}$,
(3) for any $n \in \mathbb{Z}$ and any continuous function f on the circle with $\mathrm{supp} f \subset (a_k + n\theta, a_k + (n+1)\theta)$,
$$f(u)d_{k,n} = f(u),$$
(4) for any $n \in \mathbb{Z}$ and any continuous function f on the circle with $\mathrm{supp} f \subset [a_k + n\theta, a_k + (n+1)\theta]^c$,
$$f(u)d_{k,n} = 0,$$
(5) for any $n \in \mathbb{Z}$ and any continuous function f on the circle with $\mathrm{supp} f \subset (a_k + (n+1)\theta - \frac{\theta}{4}, a_k + (n+1)\theta + \frac{\theta}{4})$,
$$(d_{k,n} + d_{k,n+1})f(u) = f(u).$$

Denote the universal C*-algebra generated by $\{\mathcal{G}' \mid \mathcal{R}'\}$ by $\mathcal{W}$, and denote by p_k a spectral projection corresponding to a half-open interval from a_k to $a_k+\theta$. It is clear that the generators $\{z, \sigma^{-n}(p_k);\ n \in \mathbb{Z}, k \in \Lambda_u\}$ of the concrete C*-algebra $\mathrm{C}(\Omega_u)$ satisfy the relations above. Moreover, let us show that these two C*-algebras are canonically isomorphic.

Lemma 2.6. *The map $u \mapsto z$, $d_{k,n} \mapsto \sigma^{-n}(p_k)$ defines a *-isomorphism from $\mathcal{W}$ to $\mathrm{C}(\Omega_u)$.*

Proof. Denote by $\phi : \mathcal{W} \to \mathrm{C}(\Omega_u)$ the map induced by $u \mapsto z$, $d_{k,n} \mapsto \sigma^{-n}(p_k)$. This surjective map is by definition a contraction. Let us show that it is an isometry.

Let π be a faithful representation of $\mathcal{W}$ on a Hilbert space $\mathscr{H}$. For any subinterval I of the circle, denote by $E(I)$ the spectral projection of $\pi(u)$ corresponding to the set I. By the relations (3) and (4) of $\mathcal{R}'$, we have that for each $k \in \mathbb{Z}$,

$$E([a_k+n\theta, a_k+(n+1)\theta]) \geq \pi(d_{k,n}) \geq E((a_k+n\theta, a_k+(n+1)\theta)).$$

Therefore,

$$\pi(d_{k,n}) = E((a_k+n\theta, a_k+(n+1)\theta)) + e_{k,n}^{(0)} + e_{k,n}^{(1)}$$

for some projections $e_{k,n}^{(0)} \leq E(\{a_k+n\theta\})$ and $e_{k,n}^{(1)} \leq E(\{a_k+(n+1)\theta\})$. Moreover, by the relation (5) of $\mathcal{R}'$, for each $n \in \mathbb{Z}$,

$$e_{k,n}^{(1)} + e_{k,n+1}^{(0)} = E(\{a_k+(n+1)\theta\}).$$

Set

$$\mathscr{H}_1 = \mathrm{span}\{e_{k,n}^{(1)}\}.$$

It is clear that $\mathscr{H}_1$ is invariant under $\mathcal{W}$ and that $e_{k,n}^{(0)} \perp \mathscr{H}_1$ for each $k \in \lambda_u, n \in \mathbb{Z}$. Denote the restrictions of π to $\mathscr{H}_1$ and $\mathscr{H}_1^{\perp}$ by π_1 and π_0 respectively. We then have that $\pi = \pi_0 \oplus \pi_1$ and

$$d_{k,n} = E_0([a_k+n\theta, a_k+(n+1)\theta)) \oplus E_1((a_k+n\theta, a_k+(n+1)\theta]),$$

where E_0 and E_1 are the spectral measures for u with respect to π_0 and π_1, respectively.

Note that the C*-algebra generated by $\{z, \sigma^n(\chi([a_k, a_k+\theta))); k \in \Lambda_u, n \in \mathbb{Z}\} \subseteq \ell^\infty(\mathbb{T})$ is isomorphic to the C*-algebra generated by $\{\bar{z}, \sigma^n(\chi((a_k, a_k+\theta])); k \in \Lambda_u, n \in \mathbb{Z}\} \subseteq \ell^\infty(\mathbb{T})$ by Lemma 2.5. It follows immediately that

$$\|a\| \leq \|\phi(a)\|$$

for any $a \in \mathcal{W}$, and so ϕ is an isometry, as desired. □

Let $\{a_k;\ k \in \Lambda_u\}$ and $\{b_l;\ l \in \Lambda_v\}$ be two sets of θ-independent real numbers. Using the lemma above, we can formulate the generators and relations for the extended rotation algebra $\mathcal{B}_\theta$ with p_k and q_l corresponding to half-open intervals from a_k to $a_k+\theta$ and from b_l to $b_l+\theta$ respectively (without specifying directions) as follows:

Lemma 2.7. *The extended rotation algebra $\mathcal{B}_\theta$ above is the universal C*-algebra generated by*

$$\mathcal{G} = \{u, p_{k,n}, v, q_{l,n};\ n \in \mathbb{Z}, k \in \Lambda_u, l \in \Lambda_v\}$$

with respect to the set of relations $\mathcal{R}$ consists of

(1) *each of* $\{u, p_{k,n};\ k \in \Lambda_u, n \in \mathbb{Z}\}$ *and* $\{v, q_{l,n};\ l \in \Lambda_v, n \in \mathbb{Z}\}$ *satisfies the set of relations* $\mathcal{R}'$ *as above,*
(2) $uv = e^{2\pi i\theta} vu$,
(3) $up_{k,n}u^* = p_{k,n-1}$, *and*
(4) $vq_{l,n}v^* = q_{l,n+1}$.

Proof. It follows from Lemma 2.6 and Definition 2.1 directly. □

Remark 2.8. From the generators and relations above, we see immediately that these extended rotation algebras do not depend on the choice of the orientation of the half-open interval. Moreover, in the case that $|\Lambda_u| = |\Lambda_v| = 1$, $a_1 = 0$ and $b_1 = 0$, if one sets $e_n = p_{1,n}$ and $f_n = q_{1,n}$, then, by the universal property, the map

$$\psi := \begin{cases} u \mapsto v^* \\ v \mapsto u \\ e_n \mapsto f_{-n-1} \\ f_n \mapsto e_n \end{cases}$$

defines an automorphism of this extended rotation algebra; that is, the order 4 automorphism of the rotation algebra induced by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ extends to an order 4 automorphism of this special extended rotation algebra.

One can go further to characterize the universal property of irrational extended rotation algebras without minimal projections by logarithms.

Theorem 2.9. *Let* $\theta \in [0,1]$ *be an irrational number, and* B *a unital* C^**-algebra. Let* Λ_u *and* Λ_v *be countable index sets (finite or infinite). If there are unitaries* u *and* v *and positive elements* $\{h_{u,k}; k \in \Lambda_u\}$ *and* $\{h_{v,l}; l \in \Lambda_v\}$ *in* B*, and real numbers* $\{a_k;\ k \in \Lambda_u\} \subseteq [0,1]$ *and* $\{b_l;\ l \in \Lambda_v\} \subseteq [0,1]$ *which are* θ*-independent such that for each* k *and* l*,*
(1) $uv = e^{2\pi i\theta} vu$,
(2) $\|h_{u,k}\| = \|h_{v,l}\| = 1$,
(3) $u = e^{2\pi i(h_{u,k}+a_k)}$, *and*
(4) $v = e^{2\pi i(h_{v,l}+b_l)}$,

then there are projections $\{p_k;\ k \in \Lambda_u\} \subseteq B$ *and* $\{q_l;\ l \in \Lambda_v\} \subseteq B$ *such that*

$$\{u, v, v^n p_k v^{-n}, u^{-n} q_l u^n;\ n \in \mathbb{Z}\}$$

satisfy the relations for the extended rotation algebra $\mathcal{B}_\theta$ *of Lemma 2.7 with cutting points* $\{a_k;\ k \in \Lambda_u\}$ *and* $\{b_l;\ l \in \Lambda_v\}$*, and* $C^*\{u, v, p_k, q_l;\ k \in \Lambda_u, l \in \Lambda_v\} = C^*\{u, v, h_{u,k}, h_{v,l};\ k \in \Lambda_u, l \in \Lambda_v\}$.

Proof. Without lose of generality, let us assume that $\theta < 1/2$.

Let π be a faithful representation of B on a Hilbert space $\mathscr{H}$, and let us use the same notation for the images of the elements of B. For each k, since $u = e^{2\pi i(h_{u,k}+a_k)}$ with $\|h_{u,k}\| = 1$, one has that

$$h_{u,k} = \eta(e^{-2\pi i a_k} u) + p_{u,k}$$

in $B(\mathscr{H})$ where $p_{u,1}$ is a subprojection of the spectral projection $E_u(\{1\})$, and η is the function on $\mathbb{T} \setminus \{e^{2\pi i a_k}\}$ sending z to $\arg(z)/2\pi$.

Note that since $v^*uv = e^{2\pi i\theta}u$, the element $v^*E_u(I)v$ is the spectral projection $E_u(\sigma(I))$, where σ is the rotation of the circle by angel θ clock-wisely.

Consider the positive element $h_1 := f_1(h_{u,k})$ and $h_2 := f_2(h_{u,k})$, where

$$f_1(x) = \begin{cases} 0 & \text{if } 0 \le x \le 1/2, \\ \text{linear} & \text{if } 1/2 \le x \le 1-\theta, \\ 1 & \text{otherwise.} \end{cases}$$

and

$$f_2(x) = \begin{cases} 1 & \text{if } 0 \le x \le \theta, \\ \text{linear} & \text{if } \theta \le x \le 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$h_1 = f_1 \circ \eta(u) + p_{u,k},$$

and

$$h_2 = f_2 \circ \eta(u) + (E_u(\{e^{2\pi i a_k}\}) - p_{u,k}).$$

Consider

$$\begin{aligned} h_1(v^*h_2v) &= (f \circ \eta(u) + p_{u,k})(v^*(f_2 \circ \eta)(u)v + v^*(E_u(\{e^{2\pi i a_k}\}) - p_{u,k})v) \\ &= ((f \circ \eta) \cdot (f_2 \circ \eta)(u) + p_{u,k} + v^*(E_u(\{e^{2\pi i a_k}\}) - p_{u,k})v \\ &= E_u((a_k - \theta, a_k)) + p_{u,k} + v^*(E_u(\{e^{2\pi i a_k}\}) - p_{u,k})v. \end{aligned}$$

It is clear that $h_1v^*h_2v \in B$ is a projection. For each integer n, set $p_{k,n} = v^{n+1}(h_1(v^*h_2v))v^{-n-1}$.

The same construction also produces projections $\{p_{k,n};\ k \in \Lambda_u, n \in \mathbb{Z}\}$ associated with the unitary u and $\{q_{l,n};\ l \in \Lambda_v, n \in \mathbb{Z}\}$ associated with the unitary v. Moreover, the unitaries u and v together with projections $\{p_{k,n};\ k \in \Lambda_u, n \in \mathbb{Z}\}$ and $\{q_{l,n};\ l \in \Lambda_v, n \in \mathbb{Z}\}$ satisfy the relations for the extended rotation algebra B_θ of Lemma 2.7 with $\{a_k;\ k \in \Lambda_u\}$ on u and $\{b_l;\ l \in \Lambda_v\}$ on v.

It is clear that $\mathrm{C}^*\{u, v, p_k, q_l;\ k \in \Lambda_u, l \in \Lambda_v\} \subseteq \mathrm{C}^*\{u, v, h_{u,k}, h_{v,k};\ k \in \Lambda_u, l \in \Lambda_v\}$. Since θ is irrational, it is not difficult to see that $h_{u,k} \in \mathrm{C}^*\{p_{k,n};\ n \in \mathbb{Z}\}$ and $h_{v,l} \in \mathrm{C}^*\{q_{l,n};\ n \in \mathbb{Z}\}$. Hence $\mathrm{C}^*\{u, v, p_k, q_l;\ k \in \Lambda_u, l \in \Lambda_v\} = \mathrm{C}^*\{u, v, h_{u,k}, h_{v,l};\ k \in \Lambda_u, l \in \Lambda_v\}$, as desired. □

Corollary 2.10. *The irrational extended rotation algebra $\mathcal{B}_\theta$ of Lemma 2.7 with cutting points $\{a_k;\ k \in \Lambda_u\}$ and $\{b_l;\ l \in \Lambda_v\}$ is the universal C*-algebra generated by the positive elements $\{h_{u,k}; k \in \Lambda_u\}$ and $\{h_{v,l}; l \in \Lambda_v\}$ and unitaries u and v with respect to the relations*

(1) $uv = e^{2\pi i\theta}vu$,
(2) $\|h_{u,k}\| = \|h_{v,l}\| = 1$,
(3) $u = e^{2\pi i(h_{u,k}+a_k)}$, *and*
(4) $v = e^{2\pi i(h_{v,l}+b_l)}$.

Proof. Denote by the universal C*-algebra of the statement by $\mathcal{B}'_\theta$. By Theorem 2.9, there is a surjective canonical *-homomorphism $\phi : \mathcal{B}_\theta \to \mathcal{B}'_\theta$. Thus, in order to prove the statement, one has to show that ϕ is also injective.

By Lemma 2.7, the C*-algebra $\mathcal{B}_\theta$ can be considered as the universal C*-algebra generated by certain concrete C*-algebras $\mathrm{C}(\Omega_u)$ and $\mathrm{C}(\Omega_v)$ (which are sub-C*-algebras of $\ell^\infty(\mathbb{T})$) with respect to the relations of Definition 2.1. Then, it is clear that there are $\{h_{u,k}; k \in \Lambda_u\}$ and $\{h_{v,l}; l \in \Lambda_v\}$ in $\mathcal{B}_\theta$ satisfying the relations of the statement, and hence there is a canonical *-homomorphism $\psi : \mathcal{B}'_\theta \to \mathcal{B}_\theta$. Moreover, by checking the proof of Theorem 2.9, one has that the generators of $\mathcal{B}_\theta$ are fixed by the composition $\psi \circ \phi$. Therefore, $\psi \circ \phi = \mathrm{id}$, and hence ϕ is injective, as desired. □

Corollary 2.11. *If there is only one cutting point, say, at $z = 1$, for each canonical unitary, the irrational extended rotation algebra $\mathcal{B}_\theta$ is the universal C*-algebra generated by unitaries u and v together with positive elements h_u and h_v satisfying the following relations:*

(1) $uv = e^{2\pi i\theta} vu$,
(2) $\|h_u\| = \|h_v\| = 1$,
(3) $u = e^{2\pi i h_u}$, *and*
(4) $v = e^{2\pi i h_v}$.

Proof. It follows from Corollary 2.10 directly. □

Remark 2.12. In Section 8, we show that the irrational extended rotation algebra $\mathcal{B}_\theta$ above is AF for generic θ, in the sense of Baire Category.

3. The unique tracial state

The irrational rotation algebra A_θ always has a unique tracial state. We shall show in this section that the irrational extended rotation algebra $\mathcal{B}_\theta$ (for arbitrary collections of subintervals $\{p_i\}$ and $\{q_j\}$ of $\mathbb{T}$) also has a unique tracial state.

For any natural numbers M and N, define a completely positive linear map $\Phi_{M,N} : \mathcal{B}_\theta \to \mathcal{B}_\theta$ by

$$\Phi_{M,N}(a) = \frac{1}{(2M+1)(2N+1)} \sum_{\substack{-M \le m \le M,\\ -N \le n \le N}} u^m v^n a v^{-n} u^{-m}.$$

The maps converge to a state:

Proposition 3.1. *For any $a \in \mathcal{B}_\theta$, the doubly indexed sequence $(\Phi_{M,N}(a))$ is Cauchy. Moreover, the limit of $\Phi_{M,N}(a)$ belongs to $\mathbb{C}\mathbf{1}$.*

Let us also introduce the following completely positive linear maps

$$\Phi^u_M(a) = \frac{1}{2M+1} \sum_{-M \le m \le M} u^m a u^{-m}$$

and

$$\Phi^v_N(a) = \frac{1}{2N+1} \sum_{-N \le n \le N} v^n a v^{-n}.$$

We then have that $\Phi_{M,N} = \Phi^u_M \circ \Phi^v_N$. Note that the maps Φ^u_M and Φ^v_N have the properties

$$\Phi^u_M(xay) = x\Phi^u_M(a)y \quad \text{for any } x, y \in \mathrm{C}(\Omega_u) \tag{3.1}$$

and

$$\Phi^v_N(xay) = x\Phi^v_N(a)y \quad \text{for any } x, y \in \mathrm{C}(\Omega_v) \tag{3.2}$$

If one considers the restriction of $\Phi_{M,N}$ to the rotation C*-algebra A_θ, then the linear maps $\{\Phi_{M,N}\}$ converge to the canonical tracial state of A_θ. Indeed, if f and g are two continuous functions on the unit circle $\mathbb{T}$, one then has

$$\begin{aligned}
\Phi_{M,N}(f(u)g(v)) &= \frac{1}{(2M+1)(2N+1)} \sum_{\substack{-M\le m\le M,\\ -N\le n\le N}} u^m v^n f(u)g(v)v^{-n}u^{-m} \\
&= \frac{1}{(2M+1)(2N+1)} \sum_{\substack{-M\le m\le M,\\ -N\le n\le N}} u^m v^n f(u)v^{-n}u^{-m}u^m v^n g(v)v^{-n}u^{-m} \\
&= \frac{1}{(2M+1)(2N+1)} \sum_{\substack{-M\le m\le M,\\ -N\le n\le N}} f(\sigma^{-n}(u))g(\sigma^m(v)) \\
&= (\frac{1}{2N+1} \sum_{-N\le n\le N} f(\sigma^{-n}(u)))(\frac{1}{2M+1} \sum_{-M\le n\le M} g(\sigma^m(v))).
\end{aligned}$$

Since θ is irrational, the functions $\frac{1}{2N+1}\sum_{-N\le n\le N} f(e^{2\pi i n\theta}z)$ converge uniformly to the constant function $\int_{\mathbb{T}} f(z)dz$ as $N \to \infty$, and hence

$$\Phi_{M,N}(f(u)g(v)) \to (\int_{\mathbb{T}} f(z)dz \int_{\mathbb{T}} g(z)dz)\mathbf{1} \quad \text{if } M, N \to \infty.$$

Since any element of A_θ can be approximated by a polynomial in elements u and v and their inverses, it follows that $\Phi_{M,N}(a)$ converges to $\tau(a)$ for any $a \in A_\theta$ where τ is the canonical tracial state of A_θ.

For a spectral projection q of v corresponding to a subinterval of the circle, since v commutes with q_j for each j and

$$u^m q_j u^{-m} = \sigma^m(q_j),$$

one has

$$\begin{aligned}
\Phi_{M,N}(q) &= \frac{1}{2M+1} \sum_{-M\le m\le M} \sigma^m(q) \\
&= \frac{1}{2M+1} \sum_{-M\le m\le M} \chi_q(e^{2\pi i m\theta} z)
\end{aligned}$$

which converges uniformly to $\mu(q)\mathbf{1}$ where μ denotes Lebesgue measure on $\mathbb{T}$ (see, for example, Section 1.1 of [Par81]). By the same argument (or by symmetry), $(\Phi_{M,N}(p))$ converges as well for a spectral projection p of u corresponding to a subinterval.

Moreover, the same calculation also shows that $(\Phi^u_N(q))$ converges to $\mu(q)\mathbf{1}$ as $N \to \infty$, and $(\Phi^v_M(p))$ converges to $\mu(p)\mathbf{1}$ as $M \to \infty$.

Consider the sub-C*-algebra of $\mathcal{B}_\theta$ generated by $\mathrm{C}(\Omega_u)$ and v. Since v acts on $\mathrm{C}(\Omega_u)$ by conjugation, by the universality of $\mathcal{B}_\theta$, this sub-C*-algebra is isomorphic to the crossed product C*-algebra $\mathrm{C}(\Omega_u)\rtimes_v\mathbb{Z}$. Note that there is a canonical conditional expectation Π_u from $\mathrm{C}(\Omega_u)\rtimes_v\mathbb{Z}$ onto $\mathrm{C}(\Omega_u)$ (see, for example, VIII.2 of [Dav96]). A direct calculation shows that for any $a\in \mathrm{C}(\Omega_u)\rtimes_v\mathbb{Z}$, the sequence $(\Phi^u_N(a))$ converges to $\Pi_u(a)$. A similar statement holds for the positive linear maps Φ^v_M.

In order to prove Proposition 3.1, one has to show that $(\Phi_{M,N}(a))$ is Cauchy for any $a\in\mathcal{B}_\theta$. Since each linear map $\Phi_{M,N}$ has norm one, it is enough to show that $(\Phi_{M,N}(a))$ is Cauchy for any word a in $\{u,u^*,v,v^*,\sigma^{-k}(p_i),\sigma^k(q_j)\}$. First, we have the following lemma, which was suggested to us by Hanfeng Li.

Lemma 3.2. *Let A and B be two unital C*-algebras, and let Φ be a unital positive linear map from A to B. Let $a\in A$ with $\|a\|=1$ be such that $\|\Phi(a^*a)\|\leq\varepsilon^2$ for some $\varepsilon>0$. Then*

$$\|\Phi(ba)\|<2\varepsilon$$

for any $b\in A$ with $\|b\|=1$.

Proof. By the Cauchy-Schwarz inequality, one has

$$\rho(\Phi(ba))\leq\varepsilon$$

for any state ρ of B. Therefore, by the polarization identity,

$$\|\Phi(ba)\|\leq 2\varepsilon,$$

as desired. □

Let us consider the map Φ^v_N. We show the following lemma.

Lemma 3.3. *For any word $a_1\cdots a_n$ where each a_i is a spectral projection of u or v, and for any $\varepsilon>0$, there exists $\{f_1,\cdots,f_n\}$ in the C*-algebra generated by $\mathrm{C}(\Omega_v)\cup\{u\}$ and a natural number K such that*

1. *f_i is a self-adjoint element in $C^*(u)$ if a_i is a spectral projection of u, and $f_i=a_i$ if a_i is a spectral projection of v, and*
2. *for any $N>K$, one has*
$$\left\|\Phi^v_N(a_i-f_i)^2\right\|<\varepsilon,$$
and
$$\|\Phi^v_N(wa_1\cdots a_n)-\Phi^v_N(wf_1\cdots f_n)\|<\varepsilon$$
for any word w of $\{u,u^*,v,v^*\}$.

Proof. Let us prove the lemma by induction on n. If $n=1$, the statement is trivial if a_1 is not a spectral projection of u. If a_1 is a spectral projection of u, let e_m denote the characteristic function corresponding to a_1. Then there is a positive real-valued continuous function f_1 on the unit circle such that

$$f_1<e_1\quad\text{and}\quad\mu(\mathrm{supp}(e_1-f_1))<\varepsilon^2/4,$$

where μ denotes Lebesgue measure. Since $e_1 - f_1 < \chi_{\mathrm{supp}(e_1-f_1)}$ and $\Phi^v_N(\chi_{\mathrm{supp}(e_1-f_1)})$ converges to $\mu(\mathrm{supp}(e_m - f_m))$, we have

$$\left\|\Phi^v_N(a_1 - f_1)^2\right\| < \varepsilon^2/4.$$

Then, by Lemma 3.2, the statement holds for $n = 1$.

Let us assume that the lemma holds for all $n \leq k$. Let us consider a word $a = a_1 \cdots a_k a_{k+1}$ in the spectral projections, and let w be any word in the canonical unitaries and their inverses. If a_{k+1} is a spectral projection of v, by Equation 3.2, we have

$$\Phi^v_N(a) = \Phi^v_N(a_1 \cdots a_k)a_{k+1}.$$

Applying the induction assumption to the word $wa_1 \cdots a_k$, we have that the statement holds for the word a.

If a_{k+1} is a spectral projection of u (therefore a_k is a spectral projection of v), there is a positive continuous function f_{k+1} on the unit circle and a natural number K such that

$$\left\|\Phi^v_N((a_{k+1} - f_{k+1}(u))^2)\right\| < (\frac{\varepsilon}{4})^2$$

for any $N > K$. Hence, by Lemma 3.2, for any $N > K$ one has

$$\|\Phi^v_N(wa_1 \cdots a_k a_{k+1}) - \Phi^v_N(wa_1 \cdots a_k f_{k+1}(u))\| < \frac{\varepsilon}{2}. \tag{3.3}$$

Without loss of generality, one may assume that $f_{k+1} = \sum_{i=-l}^{l} c_i u^i$. Then

$$\begin{aligned}\Phi^v_N(wa_1 \cdots a_k f_{k+1}(u)) &= \sum_{i=-l}^{l} c_i \Phi^v_N(wa_1 \cdots a_k u^i) \\ &= \sum_{i=-l}^{l} c_i \Phi^v_N(wa_1 \cdots a_{k-1} u^i \sigma_u^{-i}(a_k)) \\ &= \sum_{i=-l}^{l} c_i \Phi^v_N(wa_1 \cdots a_{k-1} u^i)\sigma_u^{-i}(a_k) \\ &= \sum_{i=-l}^{l} c_i \Phi^v_N(wu^i a_1^{(i)} \cdots a_{k-1}^{(i)})\sigma_u^{-i}(a_k),\end{aligned}$$

where $a_j^{(i)} = u^{-i} a_j u^i$. (Note that $a_j^{(i)} = a_j$ if a_j is a spectral projection of u.)

Applying the inductive assumption to the word $a_1 \cdots a_{k-1}$, there exist a natural number K_1 and elements $f_1, ..., f_{k-1} \in \mathrm{C}(\Omega_v) \rtimes_u \mathbb{Z}$ such that for any $1 \leq j \leq k-1$, $f_j \in \mathrm{C}^*(u)^+$ if a_j is a spectral projection of u, and $f_j = a_j$ if a_j is a spectral projection of v. Moreover, for any $N > K_1$,

$$\left\|\Phi^v_N(a_j - f_j)^2\right\| < \varepsilon, \quad 1 \leq j \leq k-1,$$

and for any $-l \leq i \leq l$,

$$\left\|\Phi^v_N(u^i wa_1 \cdots a_{k-1}) - \Phi^v_N(u^i wf_1 \cdots f_{k-1})\right\| < \frac{\varepsilon}{2(2l+1)\max\{|c_i|\}}.$$

Noting that by the construction of the map Φ^v_N, one has

$$\Phi^v_N(u^{-i}au^i) = u^{-i}\Phi^v_N(a)u^i, \quad -l \leq i \leq l.$$

Therefore, denoting by $f_j^{(i)} = u^{-i} f_j u^i$, we have

$$\begin{aligned} & \left\| \Phi_N^v(wu^i a_1^{(i)} \cdots a_{k-1}^{(i)}) - \Phi_N^v(wu^i f_1^{(i)} \cdots f_{k-1}^{(i)}) \right\| \\ = & \left\| u^i(\Phi_N^v(wu^i a_1^{(i)} \cdots a_{k-1}^{(i)}) - \Phi_N^v(wu^i f_1^{(i)} \cdots f_{k-1}^{(i)}))u^{-i} \right\| \\ = & \left\| \Phi_N^v(u^i w a_1 \cdots a_{k-1}) - \Phi_N^v(u^i w f_1 \cdots f_{k-1}) \right\| \\ \leq & \frac{\varepsilon}{2(2l+1)\max\{|c_i|\}}. \end{aligned}$$

Noting that

$$\begin{aligned} \Phi_N^v(wf_1 \cdots f_{k-1} a_k f_{k+1}(u)) &= \sum_{i=-l}^{l} c_i \Phi_N^v(wf_1 \cdots f_{k-1} a_k u^i) \\ &= \sum_{i=-l}^{l} c_i \Phi_N^v(wf_1 \cdots f_{k-1} u^i) \sigma_u^{-i}(a_k) \\ &= \sum_{i=-l}^{l} c_i \Phi_N^v(wu^i f_1^{(i)} \cdots f_{k-1}^{(i)}) \sigma_u^{-i}(a_k), \end{aligned}$$

we then have

$$\begin{aligned} & \|\Phi_N^v(wa_1 \cdots a_k f_{k+1}(u)) - \Phi_N^v(wf_1 \cdots f_{k-1} a_k f_{k+1}(u))\| \\ = & \sum_{i=-l}^{l} \left\| c_i(\Phi_N^v(wu^i a_1^{(i)} \cdots a_{k-1}^{(i)}) - \Phi_N^v(wu^i f_1^{(i)} \cdots f_{k-1}^{(i)})) \sigma_u^{-i}(a_k) \right\| \\ \leq & \varepsilon/2. \end{aligned}$$

Together with Inequality 3.3, we have

$$\|\Phi_N^v(wa_1 \cdots a_k a_{k+1}) - \Phi_N^v(wf_1 \cdots f_{k-1} a_k f_{k+1}(u))\| \leq \varepsilon.$$

Thus, the statement holds for $n = k + 1$, and therefore, the statement holds for any n, as desired. □

The lemma above also holds for the maps Φ_M^u.

Proposition 3.4. *The sequence* (Φ_N^v) *(or* (Φ_M^u)*) converges to a conditional expectation* Φ^v *(or* Φ^u*) of* $\mathcal{B}_\theta$ *onto* $C(\Omega_v)$ *(or* $C(\Omega_u)$*).*

Proof. We only give the proof for (Φ_N^v). For any $a \in \mathcal{B}_\theta$, let us show that $(\Phi_N^v(a))$ is Cauchy. We may assume that a is a monomial. By Lemma 3.3, for any $\varepsilon > 0$, there is an element $a' \in \mathrm{C}(\Omega_v) \rtimes_u \mathbb{Z} \subseteq \mathcal{B}_\theta$ and a natural number K_1 such that

$$\|\Phi_N^v(a) - \Phi_N^v(a')\| < \varepsilon/2 \quad \text{for any } N > K_1.$$

On the other hand, there is a natural number K_2 such that

$$\|\Phi_N^v(a') - \Pi_v(a')\| \leq \varepsilon/2 \quad \text{for any } N > K_2,$$

where Π_v is the canonical conditional expectation of $\mathrm{C}(\Omega_v) \rtimes_u \mathbb{Z}$ onto $\mathrm{C}(\Omega_v)$. Therefore, for any $N > \max\{K_1, K_2\}$,

$$\|\Phi_N^v(a) - \Pi_v(a')\| < \varepsilon$$

which implies that $(\Phi_N^v(a))$ is Cauchy with limit belonging to $\mathrm{C}(\Omega_v)$. Hence, (Φ_N^v) converges to a unital completely positive linear map Φ^v from $\mathcal{B}_\theta$ to $\mathrm{C}(\Omega_v)$.

By Equation 3.2, one has

$$\Phi^v(xay) = x\Phi^v(a)y$$

for any $x, y \in \mathrm{C}(\Omega_v)$. Therefore, Φ^v is a conditional expectation, as stated. □

Now, let us prove Proposition 3.1.

Proof of Proposition 3.1. Note that $\Phi_{M,N} = \Phi_M^u \circ \Phi_N^v$. Fix an element $a \in \mathcal{B}_\theta$. For any $\varepsilon > 0$, by Proposition 3.4, there are natural numbers K_1 and K_2 such that for any $N > K_1$,

$$\|\Phi_N^v(a) - \Phi^v(a)\| \leq \varepsilon/2,$$

and for any $M > K_2$,

$$\|\Phi_M^u(\Phi^v(a)) - \Phi^u(\Phi^v(a))\| \leq \varepsilon/2.$$

Then, for any $M, N > \max\{K_1, K_2\}$,

$$\begin{aligned}
\|\Phi_{M,N}(a) - \Phi^u(\Phi^v(a))\| &= \|\Phi_M^u(\Phi_N^v(a)) - \Phi^u(\Phi^v(a))\| \\
&\leq \|\Phi_M^u(\Phi_N^v(a) - \Phi^v(a))\| + \|\Phi_M^u(\Phi^v(a)) - \Phi^u(\Phi^v(a))\| \\
&\leq \varepsilon.
\end{aligned}$$

Thus, the double indexed sequence $(\Phi_{M,N})$ converges to $\Phi^u \circ \Phi^v$. Since $\Phi^v(a) \in \mathrm{C}(\Omega_v)$, we have $\Phi^u(\Phi^v(a)) \in \mathbb{C}\mathbf{1}$, as desired. □

Remark 3.5. In an earlier version of this paper, Lemma 3.3 was proved for the map $\Phi_{M,N}$ instead of Φ_M^u or Φ_N^v. Then it was pointed out by Hanfeng Li that the same argument still works for the maps Φ_M^u and Φ_N^v, and this made the original arguments for simplicity and nuclearity in Section 5 and Section 7 valid for more general extended rotation algebras.

Denote by Φ the limit of $(\Phi_{M,N})$. By Proposition 3.1, the linear map Φ is a state of $\mathcal{B}_\theta$.

Theorem 3.6. *The state Φ is the unique tracial state of $\mathcal{B}_\theta$.*

Proof. Let us first show the uniqueness. Let τ be any tracial state on $\mathcal{B}_\theta$. For any $a \in \mathcal{B}_\theta$ and any M, N, a direct calculation shows that

$$\tau(\Phi_{M,N}(a)) = \tau(a).$$

Then, by Lemma 3.1,

$$\Phi(a) = \tau(\Phi(a)) = \lim \tau(\Phi_{M,N}(a)) = \tau(a),$$

and therefore, $\Phi = \tau$.

Let us show the existence of the tracial state of $\mathcal{B}_\theta$. Consider the concrete extended rotation algebra B_θ associated with the standard GNS representation of the rotation algebra with respect to the canonical tracial state. It is well known that the von Neumann algebra A'' is a factor of type II_1. In particular, it has a tracial state. Therefore, the concrete extended rotation algebra $B_\theta \subseteq A''$ also has a tracial state. Since B_θ is a quotient of $\mathcal{B}_\theta$, this shows that $\mathcal{B}_\theta$ has a tracial state, as desired. □

By the construction, the maps $\Phi_{M,N}$, Φ^u_M, and Φ^v_N are averages of certain inner automorphisms of $\mathcal{B}_\theta$; therefore the tracial state Φ and the conditional expectations Φ^u and Φ^v preserve ideals of $\mathcal{B}_\theta$. Denote by $\mathcal{I}_\theta$ the ideal

$$\mathcal{I}_\theta := \{a \in \mathcal{B}_\theta;\ \Phi(aa^*) = 0\}.$$

Corollary 3.7. *The ideal $\mathcal{I}_\theta$ is the unique maximal proper ideal of $\mathcal{B}_\theta$. In particular, the quotient $\mathcal{B}_\theta/\mathcal{I}_\theta$ is simple.*

Proof. Let $\mathcal{I}$ be any ideal of $\mathcal{B}_\theta$. If $\mathcal{I} \not\subseteq \mathcal{I}_\theta$, then there exists $a \in \mathcal{I}$ such that

$$\Phi(aa^*) = \lambda \mathbf{1} \neq 0.$$

However, since Φ preserves ideals, one has that

$$\lambda \mathbf{1} \in \mathcal{I},$$

and hence $\mathcal{I} = \mathcal{B}_\theta$. Thus, the ideal $\mathcal{I}_\theta$ is the unique maximal proper ideal of $\mathcal{B}_\theta$. □

Recall that a state ρ of a C*-algebra A said to be faithful if $\rho(aa^*) > 0$ for any non-zero element $a \in A$.

Corollary 3.8. *Let B be any quotient C*-algebra of $\mathcal{B}_\theta$. Then the C*-algebra B has a unique tracial state $\tilde{\Phi}$. Moreover, the C*-algebra B is simple if and only if the tracial state $\tilde{\Phi}$ is faithful. In particular, the statement above holds for $B_{\pi,\theta}$ where π is a representation of A_θ.*

Proof. By Corollary 3.7, the kernel of the canonical map from $\mathcal{B}_\theta$ to B is contained in $\mathcal{I}_\theta$. Therefore, the tracial state Φ passes to a tracial state of B; denote this by $\tilde{\Phi}$. The uniqueness of tracial states of B follows from the uniqueness of tracial states of $\mathcal{B}_\theta$.

By construction, the tracial state $\tilde{\Phi}$ is faithful if and only if the kernel of the canonical kernel is $\mathcal{I}_\theta$, and if and only if B is isomorphic to $\mathcal{B}_\theta/\mathcal{I}_\theta$, and hence if and only if B is simple by Corollary 3.7. □

An immediate consequence is

Corollary 3.9. *The C*-algebra $\mathcal{B}_\theta$ is simple if and only if Φ is faithful.*

Denote by $(\pi_\tau, \mathscr{H}_\tau)$ the GNS representation of A_θ with respect to the canonical tracial state τ of A_θ. In the remaining part of the paper, let us denote the concrete extended rotation algebra $B_{\pi_\tau,\theta}$ by B_θ. Since the von Neumann algebra A''_θ is a factor of type II_1, the canonical tracial state of B_θ is faithful. By Corollary 3.8, we have the following corollary.

Corollary 3.10. *The concrete C*-algebra B_θ is simple.*

The following theorem provides a criterion for the faithfulness of the canonical tracial state.

Theorem 3.11. *Let B be a quotient C^*-algebra of $\mathcal{B}_\theta$, and denote by $\tilde{\Phi}$ the canonical tracial state of B. Let $a \in B^+$ with $\|a\| = 1$. Then $\tilde{\Phi}(a) > 0$ if and only if the following statement holds:*

There exists $\lambda > 0$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$||\sum_{i=1}^{n} u^{r_i} v^{s_i} a v^{-s_i} u^{-r_i}|| > n\lambda(1-\varepsilon)$$

for any integers $(r_i, s_i)_{i=1}^n$ with $r_i\theta, s_i\theta \in [-\delta, 0] \mod \mathbb{Z}$, where u and v are the images of the canonical unitaries of $\mathcal{B}_\theta$.

Proof. By Corollary 3.7, there is a canonical quotient map π from B to B_θ, the concrete extended rotation algebra associated with the standard representation of A_θ. If $\tilde{\Phi}(a) > 0$, then $\tau(\pi(a)) = \tilde{\Phi}(a) > 0$ where τ is the unique tracial state on B_θ. Therefore $\pi(a) \neq 0$. Let us set $\lambda = ||\pi(a)||$.

Note that in $B(\mathscr{H}_\tau)$, the sequence $(\pi(u^{r_n} v^{s_n} a v^{-s_n} u^{-r_n}))$ converges to $\pi(a)$ in the strong operator topology if $r_n\theta$ and $s_n\theta$ converge to 0 modulo $\mathbb{Z}$. Hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $r_i\theta, s_i\theta \in [-\delta, 0]$, then

$$||\sum_{i=1}^{n} \pi(u^{r_n} v^{s_n} a v^{-s_n} u^{-r_n})|| > n||\pi(a)||(1-\varepsilon).$$

Therefore, we have

$$\begin{aligned} ||\sum_{i=1}^{n} u^{r_n} v^{s_n} a v^{-s_n} u^{-r_n}|| &\geq ||\pi(\sum_{i=1}^{n} u^{r_n} v^{s_n} a v^{-s_n} u^{-r_n}||) \\ &\geq n||\pi(a)||(1-\varepsilon) \\ &= n\lambda(1-\varepsilon) \end{aligned}$$

for any integers $(r_i, s_i)_{i=1}^n$ with $r_i\theta, s_i\theta \in [-\delta, 0] \mod \mathbb{Z}$.

To prove the inverse direction of the statement, it is enough to show that $||\tilde{\Phi}_{M,N}(a)||$ has a strictly positive lower bound for sufficiently large M and N, since $\tilde{\Phi}(a) = \lim \tilde{\Phi}_{M,N}(a)$.

Suppose that for any $0 < \varepsilon < 1/2$, there exists a positive number $\delta > 0$ such that if $\{r_i, s_i\}_{i=1}^n$ satisfy $r_i\theta, s_i\theta \in [-\delta, 0] \mod \mathbb{Z}$, then

$$||\sum_{i=1}^{n} \sigma_{r_i,s_i}(a)|| > n\lambda(1-\varepsilon). \tag{3.4}$$

Define a subset R_δ of $\mathbb{Z}^2$ by

$$R_\delta(M,N) := \{(r,s);\ -M \leq r \leq M,\ -N \leq s \leq N, \text{ and } r\theta, s\theta \in [-\delta, 0] \mod \mathbb{Z}\}.$$

Since θ is an irrational number, one has

$$\lim_{M,N\to\infty} \frac{\#R_\delta(M,N)}{(2M+1)(2N+1)} = \delta^2 > 0. \tag{3.5}$$

Therefore,

$$
\begin{aligned}
||\tilde{\Phi}_{M,N}(a)|| &= ||\frac{1}{(2M+1)(2N+1)}\sum \sigma_{m,n}(a)|| \\
&\geq \frac{1}{(2M+1)(2N+1)}||\sum_{(m,n)\in R_\delta(M,N)} \sigma_{m,n}(a)|| \quad \text{(since } a \text{ is positive)} \\
&\geq \frac{\#R_\delta(M,N)}{(2M+1)(2N+1)}\lambda(1-\varepsilon) \quad \text{(by 3.4)} \\
&\geq \frac{\#R_\delta(M,N)}{2(2M+1)(2N+1)}\lambda,
\end{aligned}
$$

and hence, by 3.5,

$$\tilde{\Phi}(a) = ||\lim \tilde{\Phi}_{M,N}(a)|| \geq \lambda\delta^2/2 > 0,$$

as desired. □

A similar argument gives us the following criterion for the faithfulness of Φ^u (or Φ^v).

Theorem 3.12. *Let $a \in \mathcal{B}_\theta^+$ with $\|a\| = 1$. If there exists $\lambda > 0$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$||\sum_{i=1}^{n} v^{s_i} a v^{-s_i}|| > n\lambda(1-\varepsilon) \quad (\mathit{or} \quad ||\sum_{i=1}^{n} u^{s_i} a u^{-s_i}|| > n\lambda(1-\varepsilon))$$

for any integers $\{s_1, s_2, ..., s_n\}$ with $s_i\theta \in [-\delta, 0] \mod \mathbb{Z}$, then $\Phi^v(a) \neq 0$ (or $\Phi^u(a) \neq 0$).

4. Remarks on the nuclearity of general extended rotation algebras

In this section, we shall report some observations concerning the nuclearity of general extended rotation algebras.

Let D be a unital C*-algebra. Denote by $\mathcal{B}_\theta \dot{\otimes} D$ the algebraic tensor product of $\mathcal{B}_\theta$ and D. Denote by $\mathcal{B}_\theta \otimes_{\min} D$ and $\mathcal{B}_\theta \otimes_{\max} D$ the completion of $\mathcal{B}_\theta \dot{\otimes} D$ with respective to the minimal norm $||\cdot||_{\min}$ and the maximal norm $||\cdot||_{\max}$ respectively.

Let $\mathcal{B}_\theta \otimes D$ be the completion of $\mathcal{B}_\theta \dot{\otimes} D$ with respect to an arbitrarily given C*-norm. For any positive integers M and N, consider the completely positive linear map $\Psi^u_M : \mathcal{B}_\theta \otimes D \to \mathcal{B}_\theta \otimes D$ defined by

$$\Psi^u_M : a \mapsto \frac{1}{2M+1}\sum_{-M\leq m\leq M} U^m a U^{-m},$$

where $U = u \otimes \mathbf{1}_D$.

For any $b \otimes d \in \mathcal{B}_\theta \otimes D$, one has

$$\Psi^u_M(b \otimes d) = \Phi^u_M(b) \otimes d,$$

and hence by Proposition 3.4, the sequence $\{\Psi^u_M\}$ converges to a completely positive linear map from $\mathcal{B}_\theta \otimes D$ to $\mathrm{C}(\Omega_u) \otimes D$ (considered as a sub-C*-algebra of $\mathcal{B}_\theta \otimes D$). Denote it by Ψ^u. In particular, the argument above applies to the tensor products $\mathcal{B}_\theta \otimes_{\min} D$ and $\mathcal{B}_\theta \otimes_{\max} D$. Denote by $\Psi^u_{\min}$ and $\Psi^u_{\max}$ the corresponding completely positive linear maps. Then the diagram

$$\begin{array}{ccc} \mathcal{B}_\theta \otimes_{\min} D & \xleftarrow{\pi} & \mathcal{B}_\theta \otimes_{\max} D \\ \downarrow \Psi^u_{\min} & & \downarrow \Psi^u_{\max} \\ \mathrm{C}(\Omega_u) \otimes D & = & \mathrm{C}(\Omega_u) \otimes D \end{array}$$

commutes, where π is the canonical homomorphism from $\mathcal{B}_\theta \otimes_{\max} D$ to $\mathcal{B}_\theta \otimes_{\min} D$.

Remark 4.1. The argument above also works for the averaging by the unitary $V = v \otimes \mathbf{1}_D$ or by the unitaries U and V together. Denote the corresponding completely positive linear maps by Ψ^v, $\Psi^v_{\min}$ and $\Psi^v_{\max}$, and Ψ, $\Psi_{\min}$ and $\Psi_{\max}$, respectively. Note that $\Psi = \Psi^u \circ \Psi^v$, $\Psi_{\min} = \Psi^u_{\min} \circ \Psi^v_{\min}$, and $\Psi_{\max} = \Psi^u_{\max} \circ \Psi^v_{\max}$.

Since the C*-algebra $\mathcal{B}_\theta$ is nuclear if and only if the map π is an isomorphism for any unital C*-algebra D, with the commutative diagram above, we have the following lemma.

Lemma 4.2. *The C*-algebra $\mathcal{B}_\theta$ is nuclear if the map Ψ^u_{max} or Ψ^v_{max} is faithful for any unital C*-algebra D. In particular, if the map Ψ_{max} is faithful for any unital C*-algebra D, then the C*-algebra $\mathcal{B}_\theta$ is nuclear.*

Proof. It is sufficient to show that the canonical homomorphism π is an isomorphism for any C*-algebra D. Since the map $\Psi^u_{\max}$ is faithful, then for any non-zero element $a \in \mathcal{B}_\theta \otimes_{\max} D$, we have

$$\Psi^u_{\min}(\pi(a)\pi(a^*)) = \Psi^u_{\min}(\pi(aa^*)) = \Psi^u_{\max}(aa^*) \neq 0,$$

and hence $\pi(a) \neq 0$. Therefore, π is an isomorphism, as desired. □

By an argument similar to that of Theorem 3.11, we have the following criterion for the faithfulness of the maps Ψ, Ψ^u and Ψ^v.

Proposition 4.3. *Let D be a unital C*-algebra, and let $\mathcal{B}_\theta \otimes D$ be the completion of $\mathcal{B}_\theta \dot{\otimes} D$ with respect to an arbitrary C*-norm $||\cdot||$. Denote by Ψ, Ψ^u and Ψ^v the completely positive linear maps constructed as above. Then the following statement holds:*

Let $a \in (\mathcal{B}_\theta \otimes D)^+$ with $||a|| = 1$. If there exists $\lambda > 0$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$||\sum_{i=1}^n U^{r_i} V^{s_i} a V^{-s_i} U^{-r_i}|| \geq n\lambda(1-\varepsilon)$$

(or $||\sum_{i=1}^n U^{r_i} a U^{-r_i}|| \geq n\lambda(1-\varepsilon)$, or $||\sum_{i=1}^n V^{s_i} a V^{-s_i}|| \geq n\lambda(1-\varepsilon)$) for any integers $(r_i, s_i)_{i=1}^n$ with $r_i\theta, s_i\theta \in [-\delta, 0] \mod \mathbb{Z}$, then $\Psi(a) \neq 0$ (or $\Psi^u(a) \neq 0$, or $\Psi^v(a) \neq 0$, respectively).

Proof. The proof is similar to the proof of Theorem 3.11. □

Remark 4.4. The proposition above still holds if one replaces $[-\delta, 0]$ by $[0, \delta]$, or requires that $r_i\theta \mod \mathbb{Z}$ are in $[-\delta, 0]$ and $s_i\theta \mod \mathbb{Z}$ are in $[0, \theta]$, or $s_i\theta \mod \mathbb{Z}$ are in $[-\delta, 0]$ and $r_i\theta \mod \mathbb{Z}$ are in $[0, \theta]$.

Theorem 4.5. *If the canonical tracial state of $\mathcal{B}_\theta$ is faithful, then the completely positive linear map Ψ_{min} is faithful for any C*-algebra D.*

Proof. For any given positive element a in $\mathcal{B}_\theta \otimes_{\min} D$ with norm one, let us show that the following statement holds: For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$||\sum_{i=1}^{n} U^{r_i}V^{s_i}aV^{-s_i}U^{-r_i}||_{\min} \;\geq\; n(1-\varepsilon) \tag{4.1}$$

for any integers $(r_i, s_i)_{i=1}^n$ with $r_i\theta, s_i\theta \in [-\delta, 0] \mod \mathbb{Z}$. Then the faithfulness of $\Psi_{\min}$ follows from Proposition 4.3 (with $\lambda = 1$).

To show that (4.1) holds, it is sufficient to show that it holds for any $a \in \mathcal{B}_\theta \dot{\otimes} D$. Therefore, we may assume

$$a = \sum_{i=1}^{k} b_i \otimes d_i \in \mathcal{B}_\theta \dot{\otimes} D$$

where $b_i \in \mathcal{B}_\theta$ and $d_i \in D$, $i = 1, \cdots, k$.

Let $(\pi_D, \mathscr{H}_D)$ be a faithful representation of D, and denote by $(\pi_\tau, \mathscr{H}_\tau)$ the representation of $\mathcal{B}_\theta$ induced by the GNS representation of A_θ associated with the canonical trace. Since the canonical tracial state Φ is faithful, $\mathcal{B}_\theta$ is simple, and hence $(\pi_\tau, \mathscr{H}_\tau)$ is a faithful representation of $\mathcal{B}_\theta$. It follows that $(\pi_\tau \otimes \pi_D, \mathscr{H}_\tau \otimes \mathscr{H}_D)$ is a faithful representation of $\mathcal{B}_\theta \otimes_{\min} D$.

Note that for any $b \in \mathcal{B}_\theta$, $(\pi_\tau(u^{r_j}v^{s_j}bv^{-s_j}u^{-r_j}))$ converges to $\pi_\tau(b)$ in the strong operator topology in $B(\mathscr{H}_\tau)$ if $r_j\theta \to 0 \mod \mathbb{Z}$ and $s_j\theta \to 0 \mod \mathbb{Z}$. Thus, for any $b_i \otimes d_i$, we have that

$$\begin{aligned} \pi_\tau \otimes \pi_D(U^{r_j}V^{s_j}(b_i \otimes d_i)V^{-s_j}U^{-r_j}) &= \pi_\tau \otimes \pi_D((u^{r_j}v^{s_j}b_iv^{-s_j}u^{-r_j}) \otimes d_i) \\ &= \pi_\tau(u^{r_j}v^{s_j}b_iv^{-s_j}u^{-r_j}) \otimes \pi_D(d_i) \\ &\to \pi_\tau(b_i) \otimes \pi_D(d_i) \quad (\text{as } j \to \infty) \\ &= \pi_\tau \otimes \pi_D(b_i \otimes d_i) \end{aligned}$$

in the strong operator topology on $B(\mathscr{H}_\tau \otimes \mathscr{H}_D)$ if $r_j\theta \to 0 \mod \mathbb{Z}$ and $s_j\theta \to 0 \mod \mathbb{Z}$. Hence,

$$\pi_\tau \otimes \pi_D(U^{r_j}V^{s_j}aV^{-s_j}U^{-r_j}) \to \pi_\tau \otimes \pi_D(a) \quad (j \to \infty)$$

in the strong operator topology on $B(\mathscr{H}_\tau \otimes \mathscr{H}_D)$ if $r_j\theta \to 0 \mod \mathbb{Z}$ and $s_j\theta \to 0 \mod \mathbb{Z}$. Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} ||\pi_\tau \otimes \pi_D(\sum_{i=1}^{n} U^{r_i}V^{s_i}aV^{-s_i}U^{-r_i})|| &= ||\sum_{i=1}^{n} \pi_\tau \otimes \pi_D(U^{r_i}V^{s_i}aV^{-s_i}U^{-r_i})|| \\ &\geq n(||\pi_\tau \otimes \pi_D(a)|| - \varepsilon) \end{aligned}$$

for any integers $(r_i, s_i)_{i=1}^n$ with $r_i\theta, s_i\theta \in [-\delta, 0] \mod \mathbb{Z}$.

Since $\pi_\tau \otimes \pi_D$ is a faithful representation of $\mathcal{B}_\theta \otimes_{\min} D$, it is an isometry, and hence we have

$$||\sum_{i=1}^{n} U^{r_i}V^{s_i}aV^{-s_i}U^{-r_i}||_{\min} \geq n(1-\varepsilon).$$

This shows that (4.1) holds, and therefore, $\Psi_{\min}$ is faithful. □

With the theorem above, we have a partial converse of Lemma 4.2.

Corollary 4.6. *If the canonical tracial state Φ is faithful, then the C*-algebra $\mathcal{B}_\theta$ is nuclear if and only if the map Ψ_{max} is faithful for any C*-algebra D.*

5. The ideal structure of extended rotation algebras

In general, the universal C*-algebra $\mathcal{B}_\theta$ is not simple. For instance, if $|\Lambda_1| = 1$ and $p = E_u([-\theta, 0])$ where $E_u([a, b])$ denotes the spectral projection of u corresponding to the closed subinterval of $\mathbb{T}$ with angle from $2\pi a$ to $2\pi b$, then the element

$$(v^{-n-1}pv^{n+1})(v^{-n}pv^{n})$$

is a minimal projection which is concentrated on the singleton $\{e^{-2\pi in\theta}\}$ in the spectrum of u. These minimal projections vanish on the canonical tracial state Φ, and so the ideal $\mathcal{I}_\theta$ is non-trivial. We shall show in this section that the ideals of this type are the only nontrivial ideals appearing in extended rotation algebras.

5.1. **Simplicity of certain extended rotation algebras.** As we have seen, minimal projections (if these exist) in the commutative sub-C*-algebras $C(\Omega_u)$ and $C(\Omega_v)$ generate non-trivial ideals of $\mathcal{B}_\theta$. In this subsection, let us consider extended rotation algebras with the sub-C*-algebras $C(\Omega_u)$ and $C(\Omega_v)$ do not contain minimal projections, and show that these extended rotation algebras are simple. More precisely, for any pair of sets of θ-independent real numbers $\{a_k;\ k \in \Lambda_u\}$ and $\{b_l;\ l \in \Lambda_v\}$ with Λ_u and Λ_v countable index sets (finite or infinite), we shall consider the extended rotation algebra obtained by adding spectral projections $\{p_k;\ i \in \Lambda_u\}$ of u and spectral projections $\{q_l;\ l \in \Lambda_v\}$ of v, with each p_k corresponds to the half-open interval $[a_k, a_k + \theta)$ (or $(a_k, a_k + \theta]$) and each q_k corresponds to the half-open interval $[b_l, b_l + \theta)$ (or $(b_l, b_l + \theta]$). (By Lemma 2.5, $\mathcal{B}_\theta$ does not depend on orientation of these half-open intervals.) It is easy to see that in such a case $C(\Omega_u)$ and $C(\Omega_v)$ do not contain minimal projections. We shall prove the following theorem in this subsection.

Theorem 5.1. *The extended rotation algebra $\mathcal{B}_\theta$ generated by the spectral projections above is simple.*

We first have the following decomposition lemma for representations of extended rotation algebras.

Lemma 5.2. *With the notation above, if there is a representation of $\mathcal{B}_\theta$ on a Hilbert space $\mathscr{H}$, then there is a decomposition*

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_{-1} \oplus \mathscr{H}_1$$

such that each $\mathscr{H}_i$ ($i = -1, 0, 1$) is invariant by $\mathcal{B}_\theta$, and for each l, $E_v(\{b_l\}) = 0$ on $\mathscr{H}_0$, $q_l = E_v([b_l, b_l + \theta))$ on $\mathscr{H}_{-1}$ and $q_l = E_v((b_l, b_l + \theta])$ on $\mathscr{H}_1$.

Proof. The proof is similar to that of Lemma 3.3.1 of [MPX92]. Note that for each l and for any continuous function $f \in C(\mathbb{T})$ with $\mathrm{supp}(f) \subseteq [b_l, b_l + \theta]$, we have $f(v)q_l = q_l f(v) = f(v)$. It follows that $E_v((b_l, b_l+\theta)) \le q_l$ in $B(\mathscr{H})$. A similar argument also shows that $q_l \le E_v([b_l, b_l+\theta])$ in $B(\mathscr{H})$. Thus, we have

$$E_v((b_l, b_l + \theta)) \le q_l \le E_v([b_l, b_l + \theta]),$$

and therefore $q_l = E_v((b_l, b_l+\theta)) + q_{l,0} + q_{l,1}$ where $q_{l,0} \leq E_v(\{b_l\})$ and $q_{l,1} \leq E_v(\{b_l+\theta\})$.

Consider the eigenspaces $E_v(\{b_l\})$ and $E_v(\{b_l+\theta\})$. For any $\xi \in E_v(\{b_l\})$, we have

$$v(u^*\xi) = (vu^*v^*)v\xi = e^{2\pi i(b_l+\theta)}(u^*\xi).$$

Thus, $u^*E_v(\{b_l\}) \subseteq E_v(\{b_l+\theta\})$. A similar argument also shows that $uE_v\{b_l+\theta\} \subseteq E_v(\{b_l\})$. Therefore, the unitary u^* induces an isomorphism between $E_v(\{b_l\})$ and $E_v(\{b_l+\theta\})$.

We assert that $q_{l,0} + uq_{l,1}u^* = E_v(\{b_l\})$. Indeed, we may assume that $\theta < 1/2$ (otherwise we can consider $1-q_l$, u^*, and v^*), and thus $q_l \perp uq_lu^*$. Since $q_{l,1} \leq q_l$, we have that $q_l \perp uq_{l,1}u^*$, and in particular $q_{l,0} \perp uq_{l,1}u^*$.

If $q_{l,0} + uq_{l,1}u^* \precneqq E_v(\{b_l\})$, then there is a unit eigenvector $\xi \in \mathscr{H}$ such that $v\xi = e^{2\pi i a_k}\xi$ and $q_{l,0}\xi = uq_{l,1}u^*\xi = 0$. Hence, $q_l\xi = uq_lu^*\xi = 0$. Noting that $(q_l + uq_lu^*)f(v) = f(v)$ for any continuous function f on the unit circle with $\mathrm{supp}(f) \subseteq [b_l - \theta/2, b_l + \theta/2]$, and also noting that $f(v)\xi \in \mathbb{C}\xi$, we then have

$$f(v)\xi = (q_l + uq_lu^*)f(v)\xi = 0.$$

However, we can always choose the function f with $f(e^{2\pi i b_l}) = 1$, and then $f(v)\xi = \xi$. This is a contradiction, and hence $q_{l,0} + uq_{l,1}u^* = E_v(\{b_l\})$.

Denote by $\mathscr{H}^0$ the subspace $\overline{\mathrm{C}(\Omega_u)(\bigoplus_l E_v(\{b_l\})\mathscr{H})}$, and denote by $\mathscr{H}_0$ the orthogonal complement of $\mathscr{H}^0$. It is clear that $\mathscr{H}^0$ is invariant under $\mathrm{C}(\Omega_u)$. In order to show that it is invariant under $\mathrm{C}(\Omega_v)$, it is enough to show that it is invariant under each q_l (since $\mathrm{C}(\Omega_v)$ is generated by $\{u^nq_lu^{-n}; n \in \mathbb{Z}, l \in \Lambda_v\}$). For any $\xi \in E_v(\{b_l\})\mathscr{H}$ and any $f \in \mathrm{C}(\Omega_u)$, we have

$$v(f\xi) = vfv^*v\xi = e^{2\pi i b_l}vfv^*\xi.$$

Since $vfv^* \in \mathrm{C}(\{\Omega_u\})$, the subspace $\mathscr{H}^0$ is invariant under v, and thus invariant under $E_v((b_l, b_l+\theta))$. Moreover, for any l, since $E_v(\{b_l\})\mathscr{H} \subseteq \mathscr{H}^0$ and $E_v(\{b_l+\theta\})\mathscr{H} \subseteq \mathscr{H}^0$, one has that $q_{l,0}\mathscr{H}^0 \subseteq \mathscr{H}^0$ and $q_{l,1}\mathscr{H}^0 \subseteq \mathscr{H}^0$. Therefore $\mathscr{H}^0$ (together with it complement $\mathscr{H}_0$) is invariant under $q_l = E_v((b_l, b_l+\theta)) + q_{l,0} + q_{l,1}$, and hence under all of $\mathrm{C}(\Omega_v)$.

We shall decompose $\mathscr{H}^0$ further. Write $\mathscr{H}_{-1} = \overline{\mathrm{C}(\Omega_u)(\bigoplus_l q_{l,0}\mathscr{H})}$ and $\mathscr{H}_1 = \overline{\mathrm{C}(\Omega_u)(\bigoplus_l uq_{l,1}u^*\mathscr{H})}$. Since $E_v(\{b_l\}) = q_{l,0} + uq_{l,1}u^*$ for each l, it is clear that $\mathscr{H}^0 = \overline{\mathscr{H}_{-1} + \mathscr{H}_1}$. Let us show that $\mathscr{H}_{-1}$ is orthogonal to $\mathscr{H}_1$. It is enough to show that $q_{k,0}fq_{l,1} = 0$ for any $k, l \in \Lambda_v$ and any $f \in \mathrm{C}(\Omega_u)$. Let us show that the positive element $q_{k,0}fq_{l,1}f^*q_{k,0}$ is zero. Consider the completely positive linear map

$$\psi : \mathcal{B}_\theta \ni a \mapsto q_{k,0}aq_{k,0} \in \mathcal{B}_\theta.$$

Since $q_{k,0}u^nq_{k,0} = q_{k,0}\sigma^n(q_{k,0})u^n = 0$ for any nonzero n, with the same argument as that of Lemma 5.7 below, one has that

$$q_{k,0}f(u)q_{k,0} = (\int_{\mathbb{T}} f\mathrm{d}\mu)q_{k,0},$$

where μ is Lebesgue measure on the unit circle. Using the argument as that of Lemma 3.3, one has that for any $\varepsilon > 0$, there exists a polynomial $p(u)$ such that

$$\|q_{k,0}fq_{l,1}f^*q_{k,0} - q_{k,0}fq_{l,1}p(u)q_{k,0}\| \leq \varepsilon.$$

Since $q_{l,1}p(u)q_{k,0} = 0$ and ε is arbitrary, one has that $q_{k,0}fq_{l,1}f^*q_{k,0} = 0$, and hence $q_{k,0}fq_{l,1} = 0$. Therefore, $\mathscr{H}_{-1} \perp \mathscr{H}_1$.

For each l, let us show that $\mathscr{H}_{-1}$ and $\mathscr{H}_1$ are invariant under $q_l = E_v((b_l, b_l+\theta)) + q_{l,0} + q_{l,1}$. An argument similar to the one at the first part of the proof shows that $\mathscr{H}_{-1}$ is invariant under v, and thus invariant under $E_v((b_l, b_l+\theta))$. It is clear that $\mathscr{H}_{-1}$ is invariant under $q_{l,0}$. Since $uq_{l,1}u^*\mathscr{H}_{-1} \subseteq \mathscr{H}_1$ and $\mathscr{H}_{-1} \perp \mathscr{H}_1$, one has that $uq_{l,1}u^*\mathscr{H}_{-1} = \{0\}$, and hence $q_{l,1}\mathscr{H}_{-1} = \{0\}$. Thus, $\mathscr{H}_{-1}$ is invariant under q_l. A similar argument also shows that $\mathscr{H}_1$ also invariant under q_l.

For each l, by the construction, $E_v(\{b_l\}) = 0$ on $\mathscr{H}_0$, and thus $q_l = E_v((b_l, b_l+\theta))$. On the subspace $\mathscr{H}_{-1}$, we have that $q_{l,1} = 0$ and thus $q_l = E_v([b_l, b_l+\theta))$; on the subspace $\mathscr{H}_1$, we have that $q_{l,0} = 0$ and thus $q_l = E_v((b_l, b_l+\theta])$, as desired. □

Proposition 5.3. *With the setting above, the conditional expectation Φ^u from $\mathcal{B}_\theta$ to $C(\Omega_u)$ is faithful.*

Proof. Fix a faithful representation $(\pi, \mathscr{H})$ of $\mathcal{B}_\theta$ and a positive element $a \in \mathcal{B}_\theta^+$ with $\|a\| = 1$. By Lemma 5.2, there is a decomposition

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_{-1} \oplus \mathscr{H}_1$$

such that each $\mathscr{H}_i$ $(i = -1, 0, 1)$ is invariant under $\mathcal{B}_\theta$, and $E_v(\{b_l\}) = 0$ on $\mathscr{H}_0$, $q_l = E_v([b_l, b_l+\theta))$ on $\mathscr{H}_{-1}$ and $q_l = E_v((b_l, b_l+\theta])$ on $\mathscr{H}_1$ for any $l\Lambda_v$.

Consider the restriction of $\pi(a)$ to each invariant subspace. Since π is faithful, for any $\varepsilon > 0$, there is a unit vector ξ in one of $\mathscr{H}_i$, say $\mathscr{H}_{-1}$, such that $\|\pi(a)\xi\| \geq 1 - \varepsilon$.

Consider the cut-down of π to $\mathscr{H}_l$, and let us still denote the image of π by $\mathcal{B}_\theta$. Note that $\mathcal{B}_\theta$ is generated by $\mathrm{C}(\Omega_u)$, v, and q_l, and so there exists

$$a' = \sum_{i=1}^{n} a_{i,1}b_{i,1} \cdots a_{i,k_i}b_{i,k_i}$$

with $a_{i,j} \in \mathrm{C}(\Omega_u)$ and $b_{i,j} = v$ or q_l such that

$$\|a - a'\| \leq \varepsilon.$$

Noting that q_l is the spectral projection $E_v([b_l, b_l+\theta))$, one has that $u^{r_i}q_lu^{-r_i}$ converges to q_l in the strong operator topology if $r_i\theta$ converges to 1 from the left modulo $\mathbb{Z}$. Therefore, $u^{r_i}a'u^{-r_i}$ converges to a' in the strong operator topology if $r_i\theta$ converges to 1 from the left modulo $\mathbb{Z}$. Hence there is a $\delta > 0$ such that

$$\left\| u^r a' u^{-r}\xi - a'\xi \right\| \leq \varepsilon$$

for any r with $r\theta \in [1-\delta, 1] \mod \mathbb{Z}$.

Therefore, for any integers $r_i, r_2, ..., r_n$ with $r_i\theta \in [1-\delta, 1] \mod \mathbb{Z}$, we have

$$\begin{aligned}\left\|\sum_{i=1}^{n} u^{r_i} a u^{-r_i}\right\| &\geq \left\|\sum_{i=1}^{n} u^{r_i} a u^{-r_i}\xi\right\| \\ &\geq \left\|\sum_{i=1}^{n} u^{r_i} a' u^{-r_i}\xi\right\| - n\varepsilon \\ &\geq \left\|\sum_{i=1}^{n} a'\xi\right\| - 2n\varepsilon \\ &\geq \left\|\sum_{i=1}^{n} a\xi\right\| - 3n\varepsilon \\ &\geq n(1-\varepsilon) - 3n\varepsilon \\ &= n(1-4\varepsilon).\end{aligned}$$

By Theorem 3.12 (with $\lambda = 1$), one has that $\Phi^u(a) \neq 0$, as desired. □

Proof of Theorem 5.1. By Proposition 5.3, the conditional expectations Φ^u and Φ^v are faithful. Therefore the canonical trace $\Phi = \Phi^u \circ \Phi^v$ is faithful. By Corollary 3.9, the C*-algebra $\mathcal{B}_\theta$ is simple. □

5.2. **The ideal structure of non-simple extended rotation algebras.** Consider any extended rotation algebra $\mathcal{B}_\theta$. By Lemma 2.3, there are θ-independent real numbers $\{a_k;\ k \in \Lambda_u\}$ and $\{b_l;\ l \in \Lambda_v\}$ such that the algebra is generated by spectral projections p_k of u corresponding to $[a_k, a_k+\theta)$ or $(a_k, a_k+\theta]$, and spectral projections q_l of v corresponding to $[b_l, b_l+\theta)$ or $(b_l, b_l+\theta]$, and minimal projections (might be zero) e_k and f_l which corresponds to $\{a_k\}$ and $\{b_l\}$. Set

$$\Gamma_u = \{k \in \Lambda_u;\ e_k \neq 0\}$$

and

$$\Gamma_v = \{l \in \Lambda_v;\ f_l \neq 0\}.$$

If $\Gamma_u = \Gamma_v = \emptyset$, by Theorem 5.1, the C*-algebra $\mathcal{B}_\theta$ is simple. Otherwise, the C*-algebra $\mathcal{B}_\theta$ is non-simple, since the non-zero minimal projections have zero traces.

Lemma 5.4. *If the C*-algebra $\mathcal{B}_\theta$ is non-simple, then the unique maximal ideal $\mathcal{I}_\theta$ of $\mathcal{B}_\theta$ is generated by its minimal (non-zero) projections.*

Proof. By Lemma 2.3, the C*-algebra $\mathcal{B}_\theta$ is generated by spectral projections corresponding to half-open intervals and minimal projections. Denote by $\mathcal{I}'_\theta$ the ideal generated by all the non-zero minimal projections. Then the quotient C*-algebra $\mathcal{B}_\theta/\mathcal{I}'_\theta$ is generated by the image of the spectral projections corresponding to half-open intervals. Note that these spectral projections satisfy the relations for extended rotation algebras. Thus, by Theorem 5.1, the quotient C*-algebra $\mathcal{B}_\theta/\mathcal{I}'_\theta$ is simple, and thus $\mathcal{I}'_\theta$ is a maximal ideal of $\mathcal{B}_\theta$.

However, by Corollary 3.7, the ideal

$$\mathcal{I}_\theta = \{a \in \mathcal{B}_\theta;\ \Phi(a^*a) = 0\}$$

is the unique maximal ideal. Therefore, we conclude that $\mathcal{I}_\theta = \mathcal{I}'_\theta$, as desired. □

Remark 5.5. From the proof, we have the following short exact sequence:

$$0 \longrightarrow \mathcal{I}_\theta \longrightarrow \mathcal{B}_\theta \xrightarrow{\pi} \mathcal{B}'_\theta \longrightarrow 0,$$

where $\mathcal{B}'_\theta$ is an extended rotation algebra generated by spectral projections $\{p_k, k \in \Lambda_u\}$ and $\{q_l, l \in \Lambda_v\}$. Moreover, there is a canonical embedding $\iota\colon \mathcal{B}'_\theta \to \mathcal{B}_\theta$ such that $\pi \circ \iota = \mathrm{id}_{\mathcal{B}'_\theta}$; that is, this short exact sequence splits.

Remark 5.6. Since the minimal projection with angle $a_k + m\theta$ is unitarily equivalent to the minimal projection with angle $a_k + n\theta$, the unique maximal ideal $\mathcal{I}_\theta$ of $\mathcal{B}_\theta$ contains $|\Gamma_u| + |\Gamma_v|$ copies of the C*-algebra of compact operators.

Recall that $\mathrm{C}(\Omega_u)$ and $\mathrm{C}(\Omega_v)$ are the sub-C*-algebras of $\mathcal{B}_\theta$ generated by $\{u, v^n p_k v^{-n}, v^n e_k v^{-n};\ k \in \Lambda_u, n \in \mathbb{Z}\}$ and $\{v, u^n q_l u^{-n}, v^n f_l v^{-n};\ l \in \Lambda_v, n \in \mathbb{Z}\}$ respectively. We have the following lemma.

Lemma 5.7. *If p is a minimal projection in $C(\Omega_u)$ (or $C(\Omega_v)$), then*

$$pfp = (\int_{\mathbb{T}} f d\mu) p$$

for any $f \in C(\Omega_v)$ (or $f \in C(\Omega_u)$), where μ is Lebesgue measure on $\mathbb{T}$ (consider f as a measurable function on $\mathbb{T}$).

Proof. Without loss of generality, let us assume that p is a minimal projection in $\mathrm{C}(\Omega_u)$. Since θ is irrational, for any $n \in \mathbb{Z} \setminus \{0\}$, we have that $p \perp \sigma^n(p)$, and therefore,

$$pv^n p = pv^n p v^{-n} v^n = p\sigma^n(p) v^n = 0.$$

Hence,

$$pf(v)p = 0 \quad \text{for any } f \in \mathrm{C}(\mathbb{T}) \text{ with } \int f d\mu = 0.$$

For any $f \in \mathrm{C}(\mathbb{T})$, we have

$$0 = p(f(v) - \int f d\mu)p = pf(v)p - (\int f d\mu)p,$$

and therefore

$$pf(v)p = (\int f d\mu)p \quad \text{for any } f \in \mathrm{C}(\mathbb{T}).$$

For any spectral projection q in $\mathrm{C}(\Omega_v)$ and any $\varepsilon > 0$, there are two positive continuous functions f_0 and f_1 on $\mathbb{T}$ such that

$$f_{-1}(v) \le q \le f_1(v) \quad \text{and} \quad 0 \le \int (f_1 - f_{-1}) d\mu \le \varepsilon.$$

Hence, we have

$$(\int f_{-1} d\mu)p \le pqp \le (\int f_1 d\mu)p \quad \text{and} \quad |\int f_{-1} d\mu - \int f_1 d\mu| \le \varepsilon,$$

and therefore $pqp = (\int q d\mu)p$. Since $\mathrm{C}(\Omega_v)$ is generated by projections, we have

$$pfp = \int_{\mathbb{T}} f d\mu p \quad \text{for any } f \in \mathrm{C}(\Omega_v),$$

as desired. □

Corollary 5.8. *For any pair of minimal projections p in $C(\Omega_u)$ and q in $C(\Omega_v)$, one has that $p \perp q$.*

Proof. By the lemma above, one has

$$pqp = (\int q d\mu)p = 0,$$

and hence $p \perp q$. $\square$

Theorem 5.9. *Let p be a minimal projection in $C(\Omega_u)$ or in $C(\Omega_v)$. The hereditary sub-C*-algebra $p\mathcal{B}_\theta p$ is isomorphic to $\mathbb{C}$.*

Proof. Let us assume that $p \in \mathrm{C}(\Omega_u)$. Consider the unital completely positive linear map $\Psi : \mathcal{B}_\theta \to p\mathcal{B}_\theta p$ defined by

$$\Psi : a \mapsto pap.$$

From Lemma 5.7, it follows that $\Psi(a) \in \mathbb{C}p$ for any $a \in A_\theta \subset \mathcal{B}_\theta$.

We shall use an argument similar to that of Lemma 3.3 to show that

$$\Psi(a) \in \mathbb{C}p$$

for any word $a = a_1 \cdots a_n b_1 \cdots b_m$, where $a_i \in \{u, v, u^*, v^*\}$ and each b_i is a spectral projection of u or v. Assume that b_m is a spectral projection of v. Then, by Lemma 5.7, for any $\varepsilon > 0$, there is a positive function f_m on $\mathbb{T}$ such that

$$\left\|\Psi((b_m - f_m(v))^2)\right\| \leq (\frac{\varepsilon}{2m})^2.$$

Therefore, by Lemma 3.2, we have

$$||\Psi(a_1 \cdots a_n b_1 \cdots b_{m-1} b_m) - \Psi(a_1 \cdots a_n b_1 \cdots b_{m-1} f_m(v))|| \leq \frac{\varepsilon}{m}.$$

Without loss of generality, we may assume that $f_m(v) = \sum_{i=-l}^{l} c_i v^i$. Then we have

$$\begin{aligned} \Psi(a_1 \cdots a_n b_1 \cdots b_{m-1} f_m(v)) &= \sum_{i=-l}^{l} c_i \Psi(a_1 \cdots a_n b_1 \cdots b_{m-1} v^i) \\ &= \sum_{i=-l}^{l} c_i \Psi(a_1 \cdots a_n b_1 \cdots v^i \sigma^{-i}(b_{m-1})) \end{aligned}$$

where σ is the canonical action on $\mathrm{C}(\Omega_u)$. Since Ψ is a point evaluation on $\mathrm{C}(\Omega_u)$, there is a positive function f_{m-1} on $\mathbb{T}$ such that for any $-l \leq i \leq l$, one has

$$||\Psi(a_1 \cdots a_n b_1 \cdots u^i \sigma^{-i}(b_{m-1})) - \Psi(a_1 \cdots a_n b_1 \cdots u^i \sigma^{-i}(f_{m-1}(v)))|| \leq \frac{\varepsilon}{m(2l+1)|c_i| + 1},$$

and hence

$$\begin{aligned}
&||\Psi(a_1\cdots a_n b_1\cdots b_{m-1} f_m v) - \Psi(a_1\cdots a_n b_1\cdots f_{m-1}(u) f_m v)|| \\
= \ & ||\sum_{i=-l}^{l} c_i \Psi(a_1\cdots a_n b_1\cdots v^i\sigma^{-i}(b_{m-1})) - \sum_{i=-l}^{l} c_i\Psi(a_1\cdots a_n b_1\cdots v^i\sigma^{-i}(f_{m-1}(v)))|| \\
= \ & ||\sum_{i=-l}^{l} c_i(\Psi(a_1\cdots a_n b_1\cdots v^i\sigma^{-i}(b_{m-1})) - \Psi(a_1\cdots a_n b_1\cdots v^i\sigma^{-i}(f_{m-1}(v))))|| \\
\leq \ & \frac{\varepsilon}{m}.
\end{aligned}$$

Repeating the argument above, we obtain positive functions $f_1, \cdots, f_m$ on $\mathbb{T}$ such that

$$||\Psi(a_1\cdots a_n b_1\cdots b_i f_{i+1}(u))\cdots f_m(v) - \Psi(a_1\cdots a_n b_1\cdots f_i(v) f_{i+1}(u))\cdots f_m(v))|| \leq \frac{\varepsilon}{m}.$$

Therefore, we have

$$||\Psi(a_1\cdots a_n b_1\cdots b_m) - \Psi(a_1\cdots a_n f_1(u)\cdots f_m(v))|| \leq \varepsilon.$$

Noting that $\Psi(a_1\cdots a_n f_1(u)\cdots f_m(v)) \in \mathbb{C}p$ and that ε is arbitrary, we have

$$\Psi(a) \in \mathbb{C}p.$$

Hence $\Psi(\mathcal{B}_\theta) = \mathbb{C}p$. Noting that $\Psi : \mathcal{B}_\theta \to p\mathcal{B}_\theta p$ is surjective, we have that $p\mathcal{B}_\theta p$ is isomorphic to $\mathbb{C}$, as desired. □

For any $k \in \Gamma_u$ and $l \in \Gamma_v$, denote by $\mathcal{I}_{u,k}$ and $\mathcal{I}_{v,l}$ the ideals of $\mathcal{B}_\theta$ generated by e_k and f_l respectively. Let $\mathcal{K}$ denote the C*-algebra of compact operators on a separable Hilbert space.

Corollary 5.10. *The ideals $\mathcal{I}_{u,k}$ and $\mathcal{I}_{v,l}$ are isomorphic to $\mathcal{K}$.*

Proof. Since the ideal $\mathcal{I}_{u,k}$ is generated by the minimal projection e_k, by Brown's theorem ([Bro77]), $\mathcal{I}_{u,k}$ is stably isomorphic to $e_k\mathcal{B}_\theta e_k$. By Corollary 5.9, the C*-algebra $e_k\mathcal{B}_\theta e_k$ is isomorphic to $\mathbb{C}$, and hence $\mathcal{I}_{u,k}$ is isomorphic to $\mathcal{K}$. Similar arguments apply to the other ideals. □

Corollary 5.11. *The ideals $\mathcal{I}_{u,k}$, $\mathcal{I}_{v,l}$, $k \in \Gamma_u$, $l \in \Gamma_v$ are orthogonal to each other.*

Proof. For any pair $i, j \in \Gamma_u$ with $i \neq j$ and any element $a \in \mathcal{B}_\theta$, consider the element $e_i a e_j$. From the argument of Theorem 5.9, the element

$$(e_i a e_j)(e_j a^* e_i) = e_i a e_j a^* e_i$$

can be approximated arbitrarily close by an element in the form of $e_i a e_j b e_i$ for some b in the sub-C*-algebra A_θ. It is clear that the latter element is zero, and hence $e_i a e_j = 0$ for any $a \in \mathcal{B}_\theta$. Therefore, we have $\mathcal{I}_{u,i}\mathcal{I}_{u,j} = 0$, as desired.

Similar arguments show that the ideal $\mathcal{I}_{v,i}$ is orthogonal to the ideal $\mathcal{I}_{v,j}$ with $i, j \in \Gamma_v$ and $i \neq j$, and $\mathcal{I}_{u,k}$ orthogonal to $\mathcal{I}_{v,l}$. □

Let $\{\alpha_k;\ k \in \Gamma_u\}$ and $\{\beta_l;\ l \in \Gamma_v\}$ be a collection of strictly positive numbers such that $\sum_{k\in\Gamma_u} \alpha_k e_k$ and $\sum_{l\in\Gamma_v} \beta_l f_l$ converge in norm. Then we have the following corollary.

Corollary 5.12. *The hereditary sub-C*-algebra of $\mathcal{B}_\theta$ generated by $\sum_{k\in\Gamma_u}\alpha_k e_k+\sum_{l\in\Gamma_v}\beta_l f_l$ is isomorphic to $\bigoplus_{|\Gamma_u|+|\Gamma_v|}\mathbb{C}$.*

Proof. By Theorem 5.9 and Corollary 5.11, for any $a\in\mathcal{C}_\theta$, we have

$$(\sum_{k\in\Gamma_u}\alpha_k e_k+\sum_{l\in\Gamma_v}\beta_l f_l)a(\sum_{k\in\Gamma_u}\alpha_k e_k+\sum_{l\in\Gamma_v}\beta_l f_l)\in(\bigoplus_{k\in\Gamma_u}\mathbb{C}e_k)\oplus(\bigoplus_{l\in\Gamma_v}\mathbb{C}f_l),$$

as desired. □

Corollary 5.13. *The canonical ideal $\mathcal{I}_\theta$ of $\mathcal{B}_\theta$ is isomorphic to $\bigoplus_{|\Gamma_u|+|\Gamma_v|}\mathcal{K}$.*

Proof. By Lemma 5.4, the canonical ideal $\mathcal{I}_\theta$ is generated by the mutually orthogonal minimal projections $\{e_k, f_l;\ k\in\Gamma_u, l\in\Gamma_v\}$, and thus is generated by $\gamma=\sum_{k\in\Gamma_u}\alpha_k e_k+\sum_{l\in\Gamma_v}\beta_l f_l$. Hence by Brown's theorem, the ideal $\mathcal{I}_\theta$ is stably isomorphic to the hereditary sub-C*-algebra $\overline{\gamma\mathcal{B}_\theta\gamma}$. However, by Corollary 5.12, $\overline{\gamma\mathcal{B}_\theta\gamma}$ is isomorphic to $\bigoplus_{|\Gamma_u|+|\Gamma_v|}\mathbb{C}$, and therefore, $\mathcal{I}_\theta$ is isomorphic to $\bigoplus_{|\Gamma_u|+|\Gamma_v|}\mathcal{K}$. □

Remark 5.14. By Corollary 5.13, the short exact sequence of Remark 5.5 can be written as

$$0\longrightarrow\bigoplus_{|\Gamma_u|+|\Gamma_v|}\mathcal{K}\longrightarrow\mathcal{B}_\theta\underset{\pi}{\overset{\iota}{\leftrightarrows}}\mathcal{B}'_\theta\longrightarrow 0.$$

6. A remark on the universality of extended rotation algebras

In this section, we shall make a remark on the extended rotation algebras without non-zero minimal projections (equivalently, the simple extended rotation algebras). We shall show that any automorphism of a such irrational extended rotation algbebra is in fact determined by its restriction to the rotation algebra (Corollary 6.3).

Theorem 6.1. *Let $\mathcal{B}_\theta$ be a simple extended rotation algebra and ϕ be an endomorphism of $\mathcal{B}_\theta$ such that $\phi(u)=u$ and $\phi(v)=v$, where u and v are the canonical unitaries of $\mathcal{B}_\theta$. Then $\phi=\mathrm{id}$.*

Proof. it is enough to show that any spectral projection is fixed by ϕ. Let us consider the spectral projection p of u corresponding to an interval I. For any $\varepsilon>0$, pick continuous functions f and g in $\mathrm{C}(\mathbb{T})$ such that $f(u)p=p$, $g(u)p=g(u)$ and $\tau((f(u)-g(u))^2)\leq\varepsilon$, where τ is the canonical tracial state of B_θ. Since $\phi(u)=u$ and $\phi(v)=v$, one has

$$f(u)\phi(p)=\phi(p)\quad\text{and}\quad g(u)\phi(p)=g(u).$$

Hence

$$\begin{aligned}\phi(p)-p &= f(u)(\phi(p)-p)\\ &= (f(u)-g(u)+g(u))(\phi(p)-p)\\ &= (f(u)-g(u))(\phi(p)-p)+g(u)(\phi(p)-p)\\ &= (f(u)-g(u))(\phi(p)-p),\end{aligned}$$

and

$$\tau((\phi(p)-p)^2)\leq\tau((f(u)-g(u))^2)\leq\varepsilon.$$

Since ε is arbitrary, one has that $\tau((\phi(p)-p)^2)=0$. Since τ is faithful, one has that $\phi(p)=p$, as desired. □

Corollary 6.2. *The extended rotation algebra $\mathcal{B}_\theta$ with cutting points $\{a_k;\ k \in \Lambda_u\}$ on u and cutting points $\{b_l;\ l \in \Lambda_v\}$ on v has the following universal property: For any C*-algebra B generated by $\{u, v, h_{u,k}, h_{v,l};\ k \in \Lambda_u, l \in \Lambda_v\}$ with*

(1) $uu^* = u^*u = vv^* = v^*v = 1$,
(2) $\|h_{u,k}\| = \|h_{v,l}\| = 1$ *for each* k, l,
(3) $uv = e^{2\pi i\theta} vu$,
(4) $u = e^{2\pi i(h_{u+k}+a_k)}$, *and* $v = e^{2\pi i(h_{v+l}+b_l)}$ *for each* k, l,

there is a unique surjective homomorphism $\phi : \mathcal{B}_\theta \to B$ sending the canonical unitaries to u and v respectively.

Proof. It follows from Theorem 2.9 and Theorem 6.1. □

Corollary 6.3. *Let α and β be two automorphisms of the extended rotation algebra $\mathcal{B}_\theta$ above. If $\alpha(a) = \beta(a)$ for any $a \in A_\theta \subseteq \mathcal{B}_\theta$, then $\alpha = \beta$.*

7. Nuclearity of extended rotation algebras

In this section, we shall show that any extended rotation algebra is nuclear. Let us first consider any extended rotation algebra $\mathcal{B}_\theta$ without non-zero minimal projections. Denote by $\{a_k;\ k \in \Lambda_u\}$ and $\{b_l;\ l \in \Lambda_v\}$ the cutting points, and p_k and q_l the spectral projections corresponding to half open intervals from a_k to $a_k + \theta$ and from b_l to $b_l + \theta$ respectively.

Recall that for any C*-algebra D and any C*-algebra tensor product $\mathcal{B}_\theta \otimes D$, we have constructed completely positive linear maps Ψ^u and Ψ^v from $\mathcal{B}_\theta \otimes D$ to D in Section 4. We shall use Proposition 4.3 to show that the maps Ψ^u and Ψ^v are faithful. Hence by Lemma 4.2, the C*-algebra $\mathcal{B}_\theta$ is nuclear.

Let D be a unital C*-algebra, and let $\mathcal{B}_\theta \otimes D$ be any C*-algebra tensor product. Let π be a faithful representation of $\mathcal{B}_\theta \otimes D$ on a Hilbert space $\mathscr{H}$. Then the restriction of π to $\mathcal{B}_\theta$ or D is a representation of $\mathcal{B}_\theta$ or D on $\mathscr{H}$; denote these representations by π_1 and π_2 respectively. Note that $\pi_1(\mathcal{B}_\theta)$ and $\pi_2(D)$ commute with each other.

By Lemma 5.2, the space $\mathscr{H}$ has a decomposition

$$\mathscr{H} = \mathscr{H}_{-1} \oplus \mathscr{H}_0 \oplus \mathscr{H}_1$$

such that each $\mathscr{H}_i$ $(i = -1, 0, 1)$ is invariant under $\mathcal{B}_\theta$, and $E_v(\{b_l\}) = 0$ on $\mathscr{H}_0$, $q_l = E_v([b_l, b_l+\theta))$ on $\mathscr{H}_{-1}$, and $q_l = E_v((b_l, b_l + \theta])$ on $\mathscr{H}_1$. Let us first show that these subspaces are also invariant under D.

Lemma 7.1. *The subspaces of the decomposition*

$$\mathscr{H} = \mathscr{H}_{-1} \oplus \mathscr{H}_0 \oplus \mathscr{H}_1$$

of Lemma 5.2 with respect to the representation π_1 of $\mathcal{B}_\theta$ on $\mathscr{H}$ are invariant under D.

Proof. For convenience, let us identify the C*-algebras $\mathcal{B}_\theta$ and D with their images in $B(\mathscr{H})$, and let us use the notation of the proof of Lemma 5.2. Recall that $\mathscr{H}_0$ is defined as the orthogonal complement of the subspace

$$\mathscr{H}^0 = \overline{\mathrm{C}(\Omega_u)(\bigoplus_k E_v(\{b_l\})\mathscr{H})}.$$

For any $d \in D$, since d commutes with v, it also commutes with each $E_v(\{b_l\})$. For any $\xi \in \mathscr{H}$ and any $f \in \mathrm{C}(\Omega_u)$, since d commutes with $\mathcal{B}_\theta$, we have

$$d(f(E_v(\{b_l\})\xi)) = f(d(E_v(\{b_l\})\xi)) = f(E_v(\{b_l\})(d\xi)) \in \mathscr{H}^0.$$

Therefore, $\mathscr{H}^0$ is invariant under D, and hence $\mathscr{H}_0$ is invariant under D.

Recall the definitions

$$\mathscr{H}_{-1} = \overline{\mathrm{C}(\Omega_u)(\bigoplus_l q_{l,0}\mathscr{H})} \quad \text{and} \quad \mathscr{H}_1 = \overline{\mathrm{C}(\Omega_u)(\bigoplus_l uq_{l,1}u^*\mathscr{H})}$$

from the proof of Lemma 5.2. In order to show that $\mathscr{H}_l$ and $\mathscr{H}_r$ are invariant under D, we shall show that each projections $q_{l,0}$ and $uq_{l,1}u^*$ commute with D.

For any $d \in D$, since d commutes with the canonical unitary v, it also commutes with the spectral projections $q' = E_v([b_l, b_l + \theta/2))$ and $q'' = E_v((b_l, b_l + \theta/2))$. Recall that the projection q_l can be written as $q_l = E_v((b_l, b_l + \theta)) + q_{l,0} + q_{l,1}$, where $q_{l,0}$, and $q_{l,1}$ are subprojections of $E_v(\{b_l\})$ and $E_v(\{b_l + \theta\})$ respectively. It follows that d commutes with the element qq' and

$$qq' = (E_v((b_l, b_l + \theta)) + q_{l,0} + q_{l,1})E_v([b_l, b_l + \theta/2)) = E_v((b_l, b_l + \theta/2)) + q_0 = q'' + q_{l,0},$$

and hence d commutes with the projection $q_{l,0}$. A similar argument also shows that d commutes with the projection $q_{l,1}$. Therefore, the projections $q_{l,0}$ and $uq_{l,1}u^*$ commute with D, and the subspaces $\mathscr{H}_{-1}$ and $\mathscr{H}_1$ are invariant under D, as desired. □

We have the following immediate corollary.

Corollary 7.2. *Let π be a representation of $\mathcal{B}_\theta \otimes D$ on a Hilbert space $\mathscr{H}$. There is a decomposition*

$$\mathscr{H} = \mathscr{H}_{-1} \oplus \mathscr{H}_0 \oplus \mathscr{H}_1$$

such that each $\mathscr{H}_i$ $(i = -1, 0, 1)$ is invariant under $\mathcal{B}_\theta \otimes D$, and $E_v(\{b_l\}) = 0$ on $\mathscr{H}_0$, $q_l = E_v([b_l, b_l + \theta))$ on $\mathscr{H}_{-1}$ and $q_l = E_v((b_l, b_l + \theta])$ on $\mathscr{H}_1$.

With this corollary, let us show the main result of this section.

Proposition 7.3. *Let D be any unital C*-algebra, and $\mathcal{B}_\theta \otimes D$ the completion of the algebraic tensor product $\mathcal{B}_\theta \dot{\otimes} D$ with respect to an arbitrary pre-C*-algebra norm $||\cdot||$. Then, for any non-zero $a \in (\mathcal{B}_\theta \otimes D)^+$, one has that $\Psi^u(a) \neq 0$, where Ψ^u is the conditional expectation constructed in Section 4.*

Proof. We may assume that $||a|| = 1$. Let us show that a satisfies Proposition 4.3. That is, there exists $\lambda > 0$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$||\sum_{i=1}^{n} U^{r_i} a U^{-r_i}|| \geq n\lambda(1 - \varepsilon)$$

for any integers $\{r_i\}_{i=1}^n$ with $r_i\theta \in [-\delta, 0] \mod \mathbb{Z}$, or for any integers $\{r_i\}_{i=1}^n$ with $r_i\theta \in [0, \delta]$ mod $\mathbb{Z}$.

The proof is the same as that of Proposition 5.3. Let π be a faithful representation of $\mathcal{B}_\theta \otimes D$ on a Hilbert space $\mathscr{H}$. By Corollary 7.2, there is a decomposition

$$\mathscr{H} = \mathscr{H}_{-1} \oplus \mathscr{H}_0 \oplus \mathscr{H}_1$$

such that each subspace is invariant under $\mathcal{B}_\theta \otimes D$, and for each $l \in \Lambda_v$, $E_v(\{b_l\}) = 0$ on $\mathscr{H}_0$, $q_l = E_v([b_l, b_l + \theta))$ on $\mathscr{H}_{-1}$, and $q_l = E_v((b_l, b_l + \theta])$ on $\mathscr{H}_1$. Since π is faithful, there is a unit vector ξ in one of $\mathscr{H}_i$, say $\mathscr{H}_{-1}$, such that $\|\pi(a)\xi\| > 1 - \varepsilon$.

Consider the representation obtained from π by cutting down to the invariant subspace $\mathscr{H}_l$, and denote this by π_{-1}. Since $\pi_{-1}(q_l) = E_{\pi_{-1}(v)}([b_l, b_l + \theta))$, we have that $\pi_{-1}(u^r q_l u^{-r})$ converges to $\pi_{-1}(q_l)$ in the strong operator topology if $r\theta$ converges to 1 from the left side modulo $\mathbb{Z}$. The argument of Proposition 5.3 shows that there is a $\delta > 0$ such that

$$\left\| \pi_{-1}(U^r a U^{-r})\xi - \pi_{-1}(a)\xi \right\| \leq \varepsilon$$

for any r with $r\theta \in [1 - \delta, 1] \mod \mathbb{Z}$.

Therefore, for any integers $r_1, r_2, ..., r_n$ with $r_i\theta \in [1 - \delta, 1] \mod \mathbb{Z}$, we have

$$\begin{aligned} \left\| \sum_{i=1}^n U^{r_i} a U^{-r_i} \right\| &\geq \left\| \sum_{i=1}^n \pi_{-1}(U^{r_i} a U^{-r_i}) \right\| \\ &\geq \left\| \sum_{i=1}^n \pi_{-1}(U^{r_i} a U^{-r_i})\xi \right\| \\ &\geq n(1 - 2\varepsilon). \end{aligned}$$

Therefore, by Proposition 4.3 (with $\lambda = 1$), we have $\Psi(a) \neq 0$, as desired. □

By Lemma 4.2, we get the following theorem immediately.

Theorem 7.4. *The C*-algebra $\mathcal{B}_\theta$ is nuclear if it does not contain any non-zero minimal projection.*

Since any extended rotation algebra is an extension of an extended rotation algebra without non-zero minimal projections by the algebra of compact operators (Remark 5.14), and the property of nuclearity persists under passage to extensions, one has the following corollary.

Corollary 7.5. *Any extended rotation algebra is nuclear.*

8. Continuous field structure and AF structure

In the remaining part of the paper, we shall focus on the extended rotation algebra $\mathcal{B}_\theta$ with only one cutting point for each canonical unitary, that is, only the spectral projections $p = E_u([-\theta, 0))$ and $q = E_v([0, \theta))$ are through in. We shall show that there is a dense G_δ set $U \subseteq [0, 1]$ such that for any $\theta \in U$, the extended rotation algebra $\mathcal{B}_\theta$ is AF.

First, let us consider rational extended rotation algebras.

8.1. **Rational extended rotation algebras.** Let m and n be a pair of relatively prime natural numbers. Recall that the rational rotation algebra $A_{n/m}$ can be identified with the sub-C*-algebra of $\mathrm{M}_m(\mathrm{C}(\mathbb{T}^2))$ generated by the following two unitaries (see [Boc01]):

$$u : (z_1, z_2) \mapsto \begin{pmatrix} 0 & z_1 & \cdots & \cdots & 0 \\ 0 & 0 & z_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & z_1 \\ z_1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$v : (z_1, z_2) \mapsto \mathrm{diag}(z_2, z_2\omega, z_2\omega^2, \cdots, z_2\omega^{m-1})$$

where $\omega = e^{2\pi i n/m}$.

Consider $\mathrm{M}_m(\mathrm{C}(\mathbb{T}^2))$ as a sub-C*-algebra of the von Neumann algebra $\mathrm{M}_m(L^\infty(\mathbb{T}^2))$, and therefore consider the rotation algebra $A_{n/m}$ as a sub-C*-algebra of the von Neumann algebra $\mathrm{M}_m(L^\infty(\mathbb{T}^2))$. Denote the spectral projections $E_u((-n/m, 0])$ and $E_v([0, n/m))$ by p and q respectively. Let us consider the C*-algebra generated by $A_{n/m}$ and the spectral projections p and q in $\mathrm{M}_m(L^\infty(\mathbb{T}^2))$.

Definition 8.1. The (provisional) rational extended rotation algebra $\mathcal{B}_{n/m}$ is the sub-C*-algebra of $\mathrm{M}_m(L^\infty(\mathbb{T}^2))$ generated by u, v, p, and q.

Remark 8.2. Note that any probability measure on $\mathbb{T}^2$ induces a tracial state of the rational extended rotation algebra $\mathcal{B}_{n/m}$ by integration, and it is faithful if the support of this probability measure is $\mathbb{T}^2$.

8.2. **A field of extended rotation algebras.** Let $\theta \in [0, 1)$, and let u_θ and v_θ denote the canonical unitaries satisfying

$$u_\theta v_\theta = e^{2\pi i\theta} v_\theta u_\theta.$$

Let p_θ denote the spectral projection $E_u((-\theta, 0])$ and q_θ the spectral projection $E_v([0, \theta))$. Let us denote by $\mathcal{B}_\theta$ the extended rotation algebra generated by $\{u_\theta, v_\theta, p_\theta, q_\theta\}$. We shall construct a field of C*-algebras over the interval with fibres $\{\mathcal{B}_\theta\}_{\theta\in[0,1)}$.

Consider the families of operators $(v_\theta^k p_\theta v_\theta^{-k})$, $(u_\theta^k q_\theta u_\theta^{-k})$, (u_θ), (v_θ), and their adjoints. Then the sections

$$\sum_k f_k(\theta) a_1^{(k)} \cdots a_l^{(k)},$$

where $a_i^{(k)}$ is one of the families of operators above, and f_k is a continuous scalar-valued function on $\mathbb{T}$, form a *-algebra in a natural way. Denote it by $\mathscr{B}'$. Note that there is a evaluation map π_θ from $\mathscr{B}'$ to $\mathcal{B}_\theta$ for any θ.

Define a norm $||\cdot||$ on $\mathscr{B}'$ by

$$||\sum_k f_k(\theta)(a_1^{(k)}) \cdots (a_n^{(k)})|| := \sup_{\theta\in[0,1)} ||\pi_\theta(\sum_k f_k(\theta)(a_1^{(k)}) \cdots (a_n^{(k)}))||_{\mathcal{B}_\theta}.$$

It is easy to verify that $||\cdot||$ satisfies the C*-algebra identity. Thus, the completion of $\mathscr{B}'$ with respect to $||\cdot||$, denoted by $\mathscr{B}$, is a C*-algebra. The map $\pi_\theta : \mathscr{B}' \to \mathcal{B}_\theta$ can be extended to surjective *-homomorphism $\pi_\theta : \mathscr{B} \to \mathcal{B}_\theta$.

In this way, the C*-algebras $\mathcal{B}_\theta$ form a field of C*-algebras $(\mathcal{B}_\theta)$ over the unit circle $\mathbb{T}$. In what follows, we shall study the continuity of this field, and show that for any section $a \in \mathscr{B}$, the function $\theta \mapsto ||\pi_\theta(a)||$ is continuous at irrational points.

Remark 8.3. Note that if we only consider the subfield A of $\mathcal{B}$ generated by sections (u_θ) and (v_θ), the field A is a continuous field of C*-algebras with fibre the rotation algebra A_θ. Moreover, if we denote the canonical tracial state of A_θ by τ_θ (if θ is rational, then take τ_θ to be the integral with respect to Lebesgue measure on $\mathbb{T}^2$), then the function $\theta \mapsto \tau_\theta(\pi_\theta(a))$ is continuous for any section a of A. (See [Ell82].)

8.3. **Lower semicontinuity.** Denote by τ_θ the canonical tracial state of $\mathcal{B}_\theta$ (if θ is rational, then take τ_θ to be the integral with respect to Lebesgue measure). We then have

Lemma 8.4. *$\tau_\theta(\pi_\theta(a))$ is continuous with respect to θ for any section $a \in \mathscr{B}$.*

Proof. By linearity and continuity, we only need to prove the statement for $a = a_1 \cdots a_n$ where $a_i \in \{(u_\theta), (v_\theta), (u_\theta^*), (v_\theta^*), (p_\theta), (q_\theta)\}$. For any θ_0 and any $\varepsilon > 0$, there are continuous functions p'_{θ_0} and q'_{θ_0} defined on the unit circle, and $\delta_1 > 0$, such that if one substitutes $p'_{\theta_0}(u)$ for p_θ and $q'_{\theta_0}(u)$ for q_θ, and denotes the modified sequence $a_1, ..., a_n$ by $a'_1, ..., a'_n$, then

$$|\tau_{\theta'}(a_1 \cdots a_n) - \tau_{\theta'}(a'_1 \cdots a'_n)| < \frac{\varepsilon}{3}$$

for any $|\theta' - \theta_0| < \delta_1$.

Since $\pi_\theta(a'_i) \in \mathcal{A}_\theta$, the section $a'_1 \cdots a'_n$ is in the field of rotation algebras (A_θ), and therefore $\tau_\theta(\pi_\theta(a'_1 \cdots a'_n))$ is continuous with respect to θ by Remark 8.3. Hence there exists $\delta_2 > 0$ such that

$$|\tau_{\theta'}(a'_1 \cdots a'_n) - \tau_{\theta_0}(a'_1 \cdots a'_n)| < \frac{\varepsilon}{3}$$

if $|\theta' - \theta_0| < \delta_2$. Then for any θ' with $|\theta' - \theta_0| < \min(\delta_1, \delta_2)$, we have

$$\begin{aligned} |\tau_{\theta'}(a_1 \cdots a_n) - \tau_{\theta_0}(a_1 \cdots a_n)| &< |\tau_{\theta'}(a'_1 \cdots a'_n) - \tau_{\theta_0}(a'_1 \cdots a'_n)| + \frac{2\varepsilon}{3} \\ &< \varepsilon, \end{aligned}$$

as desired. □

Proposition 8.5. *The field $(\mathcal{B}_\theta)$ is lower semicontinuous.*

Proof. By the continuous functional calculus, it is enough to show that for any section a of $(\mathcal{B}_\theta)$, if the sequence (θ_n) converges to θ, and $\pi_{\theta_n}(a) = 0$, then $\pi_\theta(a) = 0$.

Since $\pi_{\theta_n}(a) = 0$, it follows that $\tau_{\theta_n}(\pi_{\theta_n}(aa^*)) = 0$. By Lemma 8.4,

$$\tau_\theta(\pi_\theta(a)) = \lim \tau_{\theta_n}(\pi_{\theta_n}(a)) = 0.$$

Since τ_θ is faithful on $\mathcal{B}_\theta$, we have that $\pi_\theta(aa^*) = 0$ which implies $\pi_\theta(a) = 0$. □

8.4. **Upper semicontinuity.** We shall study the upper semicontinuity of the field of C*-algebras $(\mathcal{B}_\theta)$. The sections of $(\mathcal{B}_\theta)$ are not always continuous; for example, the section

$$n1 - (\sum_{i=0}^{m-1} v_\theta^i p_\theta v_\theta^{-i})$$

is not continuous at the point n/m. However, we shall show that the field $(\mathcal{B}_\theta)$ is continuous at any irrational point.

By analogy with the construction of the field $(\mathcal{B}_\theta)$ (and in fact as a subfield of this field), let us construct an auxiliary field of commutative C*-algebras (Ω_θ^u) where Ω_θ^u is the C*-algebra generated by $\{u_\theta, v_\theta^n p_\theta v_\theta^{-n};\ n \in \mathbb{Z}\}$. (Note that the unitary u_θ is in the C*-algebra generated by $\{v_\theta^n p_\theta v_\theta^{-n}\}$ if θ is an irrational number.) It is easy to see that the spectrum of Ω_θ^u is the Cantor set if θ is irrational, and is the union of disjoint intervals if $\theta = n/m$ with $(n, m) = 1$.

The field of C*-algebras (Ω_θ^u), by definition, is generated by the sections

$$\theta \to f(\theta) a_1 a_2$$

where $a_1, a_2 \in \{(u_\theta^n)_\theta, (v_\theta^n p_\theta v_\theta^{-n})_\theta;\ n \in \mathbb{Z}\}$ and f is a continuous function on $\mathbb{T}$. Let π_θ denote the projection map from (Ω_θ^u) to Ω_θ^u.

Lemma 8.6. *The field* (Ω_θ^u) *is continuous at any irrational point.*

Proof. Let a be a section of (Ω_θ^u), and let θ_0 be a irrational point. It is enough to show that if $\pi_{\theta_0}(a) = 0$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\pi_\theta(a)\| < \varepsilon$$

for any θ with $|\theta - \theta_0| < \delta$.

Note that a can be approximated by a finite sum

$$a' = \sum_{k=1}^{K} c_k(\theta)(u_\theta^{n_k})(p_{k1}) \cdots (p_{km_k}),$$

where the c_k are continuous functions on $\mathbb{T}$, and each (p_{km_j}) is one of the sections of projections $(v_\theta^n p_\theta v_\theta^{-n})_\theta,\ n \in \mathbb{Z}$. We may assume that

$$\|a - a'\| < \varepsilon/4.$$

Since $\pi_{\theta_0}(a) = 0$, one has in particular that $\|\pi_{\theta_0}(a')\| < \varepsilon/4$.

Note that at each point θ, $\pi_\theta(a')$ can be simplified as

$$\pi_\theta(a') = \sum_{i=1}^{m_\theta}(c_{i1}(\theta)u_\theta^{n_{i1}} + \cdots + c_{ij_i}(\theta)u_\theta^{n_{ij_i}})P_i,$$

where the P_i are mutually orthogonal projections in Ω_θ^u, and the family (c_{ij}) is a certain rearrangement of the continuous functions (c_k).

Since the projection $v_\theta^k p_\theta v_\theta^{-k}$ is the spectral projection in $L^\infty(\mathbb{T})$ of the unitary u_θ corresponding to the set $[k\theta, (k+1)\theta)$, the projection P_i is a spectral projection of the unitary u_θ corresponding to the set $[m\theta, n\theta)$ for certain integers m and n. Let us consider each commutative C*-algebra Ω_θ^u

as the sub-C*-algebra of $L^\infty(\mathbb{T})$ generated by the canonical unitary and its spectral projections $\{\chi_{[k\theta,(k+1)\theta)};\ k \in \mathbb{Z}\}$.

Note that the mutually orthogonal projections $\{P_i\}$ induce a partition of the circle $\mathbb{T}$. Denote by $\{T_{i,\theta};\ i = 1, \cdots, m_\theta{+}1\}$ the partition with $T_{i,\theta}$ corresponding to P_i, and $T_{m_\theta+1,\theta}$ corresponding to the complement of the sum of P_i (if the complement is zero, then, $T_{m_\theta+1,\theta}$ is empty). Then $\pi_\theta(a')$ can be considered as a continuous function on the partition; in other words,

$$\pi_\theta(a') = (a'_{1,\theta}, \cdots, a'_{m_\theta+1,\theta}) \in \mathrm{C}([0,1]) \oplus \cdots \oplus \mathrm{C}([0,1])$$

where each copy of $[0,1]$ corresponds to a piece of the partition of $\mathbb{T}$, and $a'_{i,\theta} = c_{i1}(\theta)u^{n_{i1}} + \cdots + c_{ij_i}(\theta)u^{n_{ij_i}}$.

Since θ_0 is irrational, there exists $\delta > 0$ such that for any θ with $|\theta - \theta_0| < \delta$, one has that $m_\theta = m_{\theta_0}$, and each piece $T_{i,\theta}$ of the partition corresponding to θ only changes slightly from the piece T_{i,θ_0} (these sets are subintervals of the circle). Therefore, since u_θ corresponds to the canonical generator of $\mathrm{C}(\mathbb{T})$ which is a continuous function on the circle, and the functions c_{ij} are continuous, by choosing δ sufficiently small, we may assume that for any $|\theta - \theta_0| < \delta$,

$$\left| \left\| a'_{i,\theta} \right\| - \left\| a'_{i,\theta_0} \right\| \right| < \varepsilon/2 \quad \text{for any } i = 1, \cdots, m_\theta.$$

Hence, we have

$$\|\pi_\theta(a') - \pi_{\theta_0}(a')\| < \varepsilon/2$$

for any θ with $|\theta - \theta_0| < \delta$. Therefore,

$$\|\pi_\theta(a)\| < \|\pi_\theta(a')\| + \varepsilon/4 < \|\pi_{\theta_0}(a')\| + 3\varepsilon/4 < \varepsilon$$

holds for any θ with $|\theta - \theta_0| < \delta$, as desired. □

In the same way, the sections $\{(v_\theta), (u_\theta^n q_\theta u_\theta^{-n})\}$ generate a field of C*-algebras (Ω_θ^v). Similar arguments show that the field (Ω_θ^v) is continuous at irrational points. (Alternatively, the fields corresponding to u and v are isomorphic.)

For any irrational number θ_0, set

$$I'_{\theta_0} = \{a \in (\Omega_\theta^u); \pi_{\theta_0}(a) = 0\}$$

and

$$J'_{\theta_0} = \{b \in (\Omega_\theta^v); \pi_{\theta_0}(b) = 0\}.$$

Denote by $\mathcal{B}$ the universal C*-algebra generated by the sections of (Ω_θ^u) and (Ω_θ^v) with respect to the relations

$$(u_\theta)(b_\theta)(u_\theta^*) = (\sigma_\theta(b_\theta))$$

and

$$(v_\theta)(a_\theta)(v_\theta^*) = (\sigma_\theta^{-1}(a_\theta)),$$

where (a_θ) and (b_θ) are sections of (Ω_θ^u) and (Ω_θ^v) respectively. There exists a canonical surjective homomorphism $\psi : \mathcal{B} \to \mathscr{B}$, such that the composition map

$$\pi_{\theta_0} \circ \psi : \mathcal{B} \to \mathcal{B}_{\theta_0}$$

sends I'_{θ_0} and J'_{θ_0} to zero.

However, the quotient C*-algebra $\mathcal{B}/\mathrm{Ideal}(I'_{\theta_0}, J'_{\theta_0})$ is generated by the canonical unitaries and their spectral projections corresponding to the half-open intervals. Therefore, it is a simple C*-algebra by Theorem 5.1, and is isomorphic to $\mathcal{B}_\theta$; in particular, one has

$$\mathrm{Ideal}(I'_{\theta_0}, J'_{\theta_0}) = \ker(\pi_{\theta_0} \circ \psi).$$

Let c be a section of $(\mathcal{B}_\theta)$. If $\pi_{\theta_0}(c) = 0$, then any pre-image c' of c in $\mathcal{B}$ is in $\ker(\pi_{\theta_0} \circ \psi)$, and thus c lies inside the image of $\mathrm{Ideal}(I'_{\theta_0}, J'_{\theta_0})$. Without loss of generality, we may assume that

$$c' = \sum_{i=1}^{n} x_i a_i y_i + \sum_{j=1}^{m} g_j b_j h_j$$

where $a_i \in I'_{\theta_0}$, $b_j \in J'_{\theta_0}$, and $x_i, y_i, g_j, h_j \in \mathcal{B}$. Hence we have

$$\|\pi_\theta \circ \psi(c')\| \leq \sum_{i=1}^{n} \|x_i\| \, \|y_i\| \, \|\pi_\theta(a_i)\| + \sum_{j=1}^{m} \|g_j\| \, \|h_j\| \, \|\pi_\theta(b_j)\| .$$

Since $\pi_{\theta_0}(a_i) = 0$ and $\pi_{\theta_0}(b_j) = 0$, and the sections of (Ω^u_θ) and (Ω^v_θ) are continuous at irrational points by Lemma 8.6, we have that for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|\pi_\theta \circ \psi(c')\| < \varepsilon \quad \text{for any } \theta \text{ with } |\theta - \theta_0| < \delta.$$

In particular, one has

$$\|\pi_\theta(c)\| < \varepsilon \quad \text{for any } \theta \text{ with } |\theta - \theta_0| < \delta.$$

By the continuous functional calculus, the field $(\mathcal{B}_\theta)$ is upper semicontinuous at θ_0.

Proposition 8.7. *The field $(\mathcal{B}_\theta)$ is upper semicontinuous at irrational points.*

Together with Proposition 8.5, we have the following theorem.

Theorem 8.8. *The field of C*-algebras $(\mathcal{B}_\theta)$ is continuous at irrational points.*

8.5. **An upper semicontinuous field of C*-algebras.** Since the field $(\mathcal{B}_\theta)$ constructed in the last subsection is not upper semicontinuous at rational points, we cannot get any information on the structure and the K-theory of the irrational extended rotation algebras from the fibres at rational points. In this subsection, we shall construct new rational fibres, to obtain a new field of C*-algebras, still denoted by $(\mathcal{B}_\theta)$—we trust that, since the irrational fibres will be exactly the same, there will be no confusion—, which is upper semicontinuous. In the remaining part of the paper, we shall show that the rational fibres of this field are approximate dimension drop circle algebras, and hence using the upper semicontinuous field structure and the classification theorem for simple inductive limits of dimension drop circle algebras, we shall show that for generic θ, the C*-algebra $\mathcal{B}_\theta$ is an AF-algebra with

$$(\mathrm{K}_0(\mathcal{B}_\theta), \mathrm{K}_0^+(\mathcal{B}_\theta), [\mathbf{1}_{\mathcal{B}_\theta}]) \cong (\mathbb{Z} + \theta\mathbb{Z}, (\mathbb{Z} + \theta\mathbb{Z}) \cap \mathbb{R}^+, 1).$$

Recall that the fields (Ω^u_θ) and (Ω^v_θ) of commutative C*-algebras are continuous at irrational points, and not continuous at rational points (even not upper semicontinuous). For each $\theta \in [0, 1]$, let us define

$$I_\theta := \{a \in (\Omega^u_\theta);\ \lim_{\theta' \to \theta} (\pi_{\theta'}(a)) = 0\}$$

and

$$J_\theta := \{b \in (\Omega_\theta^v);\ \lim_{\theta' \to \theta} (\pi_{\theta'}(b)) = 0\}.$$

Denote by Ω^u and Ω^v the C*-algebras of the fields (Ω_θ^u) and (Ω_θ^v) respectively. Then, for any θ, consider the C*-algebras

$$\tilde{\Omega}_\theta^u = \Omega^u / I_\theta$$

and

$$\tilde{\Omega}_\theta^v = \Omega^v / J_\theta.$$

Remark 8.9. By Lemma 8.6, one has that $\tilde{\Omega}_\theta^u = \Omega_\theta^u$ and $\tilde{\Omega}_\theta^v = \Omega_\theta^v$ for irrational θ. However, for rational θ, the C*-algebras Ω_θ^u and Ω_θ^v are only quotient C*-algebras of $\tilde{\Omega}_\theta^u$ and $\tilde{\Omega}_\theta^v$ respectively.

Remark 8.10. The C*-algebra $\tilde{\Omega}_\theta^u$ is generated by the images of the sections

$$\{(u_\theta), (v_\theta)^k (p_\theta)(v_\theta)^{-k};\ k \in \mathbb{Z}\}.$$

In the sequel, let us denote by u and p_k the images of (u_θ) and $(v_\theta)^k (p_\theta)(v_\theta)^{-k}$ in $\tilde{\Omega}_\theta^u$, and by v and q_k denote the images of (v_θ) and $(u_\theta)^k (q_\theta)(u_\theta)^{-k}$ in $\tilde{\Omega}_\theta^v$, if there is no confusion with the C*-algebras of sections generated by u and p_k and by v and q_k, respectively.

Consider the fields of C*-algebra with $(\tilde{\Omega}_\theta^u)$ and $(\tilde{\Omega}_\theta^v)$. We then have the following lemma.

Lemma 8.11. *The fields of C*-algebras $(\tilde{\Omega}_\theta^u)$ and $(\tilde{\Omega}_\theta^v)$ are upper semicontinuous.*

Proof. For any $c \in \Omega^u$, if $\pi_\theta(c) = 0$, then $c \in I_\theta$. By the construction of I_θ, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|\pi_{\theta'}(c)\| \leq \varepsilon$ for any θ' with $|\theta' - \theta| \leq \delta$. By the continuous functional calculus, it follows that the field $(\tilde{\Omega}_\theta^u)$ is upper semicontinuous. The argument same as above applies to the field $(\tilde{\Omega}_\theta^v)$. □

Definition 8.12. For each θ, denote by $\mathcal{B}_\theta$ the universal C*-algebra generated by $\tilde{\Omega}_\theta^u$ and $\tilde{\Omega}_\theta^v$ with the relations

(1) $uv = e^{2\pi i\theta} vu$,
(2) $vp_k v^* = p_{k+1}$, and
(3) $uq_k u^* = q_{k+1}$.

Remark 8.13. It is clear that for irrational θ, the C*-algebra $\mathcal{B}_\theta$ defined above is exactly the irrational extended rotation algebra. Therefore, we use the same notation. But for θ rational, the algebra is different from the rational extended rotation algebra defined in 8.1. (We use the same notation because the present choose of fibre seems much more appropriate.)

Definition 8.14. For each θ, denote by $\mathcal{E}_\theta^u$ the universal C*-algebra generated by $\tilde{\Omega}_\theta^u$ and v with the relations

(1) $vv^* = v^*v = 1$,
(2) $uv = e^{2\pi i\theta} vu$, and
(3) $vp_k v^* = p_{k+1}$,

and denote by $\mathcal{E}_\theta^v$ the universal C*-algebra generated by $\tilde{\Omega}_\theta^v$ and u with the relations

(1) $uu^* = u^*u = 1$,

(2) $uv = e^{2\pi i\theta} vu$, and
(3) $uq_k u^* = q_{k+1}$.

Remark 8.15. By definition, the C*-algebra $\mathcal{B}_\theta$ is canonically isomorphic to the amalgamated free product $\mathcal{E}^u_\theta *_{A_\theta} \mathcal{E}^v_\theta$. (See [Tho03]. We are indebted to Hanfeng Li for this observation; cf. Remark 8.28 below.)

Consider the field of C*-algebras $(\mathcal{B}_\theta)$ with sections $\{(u_\theta), (p_k), (v_\theta), (q_k); k \in \mathbb{Z}\}$, or, rather, with the *-algebra of sections generated by these.

Theorem 8.16. *The field of C*-algebras $(\mathcal{B}_\theta)$ is upper semicontinuous.*

Proof. Denote by $\mathscr{B}$ the C*-algebra of $(\mathcal{B}_\theta)$. Consider the universal C*-algebra $\mathcal{B}$ generated by the fundamental sections of $(\tilde{\Omega}^u_\theta)$ and $(\tilde{\Omega}^v_\theta)$ with the relations

(1) $(u_\theta)(v_\theta) = e^{2\pi i\theta}(v_\theta)(u_\theta)$,
(2) $(v_\theta)(p_k)(v_\theta)^* = (p_{k+1})$, and
(3) $(u_\theta)(q_k)(u_\theta)^* = (q_{k+1})$.

Then, there is a surjective map $\psi : \mathcal{B} \to \mathscr{B}$ sending each generator of $\mathcal{B}$ to the corresponding element of $\mathscr{B}$.

Fix θ_0. Set

$$I'_{\theta_0} := \{a \in (\tilde{\Omega}^u_\theta);\ \pi_{\theta_0}(a) = 0\}$$

and

$$J'_{\theta_0} := \{b \in (\tilde{\Omega}^v_\theta);\ \pi_{\theta_0}(b) = 0\}.$$

Consider the map $\pi_{\theta_0} \circ \psi : \mathcal{B} \to \mathcal{B}_{\theta_0}$. It is clear that

$$\mathrm{Ideal}(I'_{\theta_0}, J'_{\theta_0}) \subseteq \ker(\pi_{\theta_0} \circ \psi).$$

On the other hand, the C*-algebra $\mathcal{B}/\mathrm{Ideal}(I'_{\theta_0}, J'_{\theta_0})$ is generated by $\{u_{\theta_0}, p_k, v_{\theta_0}, q_k;\ k \in \mathbb{Z}\}$ with the relations of Definition 8.12. Therefore, by the universality of $\mathcal{B}_{\theta_0}$, we have

$$\mathrm{Ideal}(I'_{\theta_0}, J'_{\theta_0}) = \ker(\pi_{\theta_0} \circ \psi).$$

Thus, if c is a section in $(\mathcal{B}_\theta)$ with $\pi_{\theta_0}(c) = 0$, then $c = \psi(c')$ for some $c' \in \mathrm{Ideal}(I'_{\theta_0}, J'_{\theta_0})$. Therefore, without loss of generality, we may assume that

$$c' = \sum_{i=1}^{n} x_i a_i y_i + \sum_{j=1}^{m} g_j b_j h_j$$

where $a_i \in I'_{\theta_0}$, $b_j \in J'_{\theta_0}$, and $x_i, y_i, g_j, h_j \in \mathcal{B}$. Hence we have

$$\|\pi_\theta \circ \psi(c')\| \leq \sum_{i=1}^{n} \|x_i\| \|y_i\| \|\pi_\theta(a_i)\| + \sum_{j=1}^{m} \|g_j\| \|h_j\| \|\pi_\theta(b_j)\|.$$

Since $\pi_{\theta_0}(a_i) = 0$ and $\pi_{\theta_0}(b_j) = 0$, and the sections of $(\tilde{\Omega}^u_\theta)$ and $(\tilde{\Omega}^v_\theta)$ are upper semicontinuous by Lemma 8.11, we have that for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|\pi_\theta \circ \psi(c')\| < \varepsilon \quad \text{for any } \theta \text{ with } |\theta - \theta_0| < \delta.$$

In particular, one has

$$\|\pi_\theta(c)\| < \varepsilon \quad \text{for any } \theta \text{ with } |\theta - \theta_0| < \delta,$$

as desired. □

8.6. **Structure of the rational fibres.** By Remark 8.13, we have that $(\mathcal{B}_\theta)$ has the irrational extended rotation algebra as irrational fibre. For a rational n/m with $(n, m) = 1$, the fibre C*-algebra $\mathcal{B}_{n/m}$ is not the rational extended rotation algebra defined by 8.1. Let us consider the ideal of $\mathcal{B}_{n/m}$ generated by $\{p^{(1)}, p^{(-1)}, q^{(1)}, q^{(-1)}\}$, where

$$p^{(1)} = p_0(v^{m+1}p_0v^{-m-1}), \quad p^{(-1)} = p_0(v^{m-1}p_0v^{-m+1}),$$

and

$$q^{(1)} = q_0(u^{m+1}q_0u^{-m-1}), \quad q^{(-1)} = q_0(u^{m-1}q_0u^{-m+1}).$$

Lemma 8.17. *The projections $p^{(1)}$ and $p^{(-1)}$ have the properties*

1. *for any $k \neq 0$, $(v^kp^{(1)}v^{-k})p^{(1)} = 0$,*
2. *for any $k \neq 0$, $(v^kp^{(-1)}v^{-k})p^{(-1)} = 0$,*
3. *for any $k \in \mathbb{Z}$, $(v^kp^{(1)}v^{-k})p^{(-1)} = 0$,*
4. *for any continuous function $f \in \mathrm{C}(\mathbb{T})$ with $f(e^{2\pi in/m}) = 0$ one has that $p^{(1)}f(u) = 0$, and*
5. *for any continuous function $f \in \mathrm{C}(\mathbb{T})$ with $f(1) = 0$ one has that $p^{(-1)}f(u) = 0$.*

Proof. We only prove Property 1, Property 3, and Property 4. Property 2 and Property 5 can be proved in a similarly way.

Consider the section $(p^{(1)}) = (p_\theta)(v_\theta^{m+1}p_\theta v_\theta^{-m-1}) \in (\Omega_\theta^u)$. Then $p^{(1)}$ is the image of $(p^{(1)})$ in $\tilde{\Omega}_{n/m}^u$. Consider the section $(v_\theta^{-k})(p^{(1)})(v_\theta^k)(p^{(1)})$ consisting of the spectral projection corresponding to the interval

$$[(m+1)\theta, \theta) \cap [(m+1)\theta + k\theta, \theta + k\theta),$$

where $[a, b)$ is empty if $a > b \mod \mathbb{Z}$. Noting that this half-open interval on the circle is eventually empty if $\theta \to n/m$, one sees

$$(v_\theta^{-k})(p^{(1)})(v_\theta^k)(p^{(1)}) \in I_{n/m}$$

for any $k \neq 0$, and therefore Property 1 holds.

Since the projection $p^{(1)}$ comes from the section of spectral projections $[(m+1)\theta, \theta)$, one has that $gp^{(1)} = p^{(1)}$ for any continuous function $g \in \mathrm{C}(\mathbb{T})$ with $g(e^{2\pi ti}) = 1$ for any $t \in [n/m - \delta, n/m + \delta]$ for some $\delta > 0$. Since $f(e^{2\pi in/m}) = 0$, for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(e^{2\pi ti})| < \varepsilon$ for any $t \in [n/m - \delta, n/m + \delta]$. Choosing $g \in \mathrm{C}(\mathbb{T})$ with $g(e^{2\pi ti}) = 1$ for any $t \in [n/m - \delta/2, n/m + \delta/2]$ and $g(e^{2\pi ti}) = 0$ for any t in the complement of $(n/m - \delta, n/m + \delta)$, we then have

$$\left\|f(u)p^{(1)}\right\| = \left\|f(u)g(u)p^{(1)}\right\| \leq \|f(u)g(u)\| \left\|p^{(1)}\right\| < \varepsilon.$$

Since ε is arbitrary, one has that $f(u)p^{(1)} = 0$, and so Property 4 holds.

To verify Property 3, it is enough to note that the fibre of the section $(p^{(1)}) = (p_\theta)(v_\theta^{m+1}p_\theta v_\theta^{-m-1})$ is eventually zero if $\theta \to n/m$ counter-clockwise, and the fibre of the section $(p^{(-1)}) = (p_\theta)(v_\theta^{m-1}p_\theta v_\theta^{-m+1})$ is eventually zero if $\theta \to n/m$ clockwise. □

Consider the exact sequences

$$0 \longrightarrow \mathrm{Ideal}\{p^{(1)}, p^{(-1)}, q^{(1)}, q^{(-1)}\} \longrightarrow \mathcal{B}_{n/m} \longrightarrow \mathcal{B}'_{n/m} \longrightarrow 0, \tag{8.1}$$

$$0 \longrightarrow \mathrm{Ideal}\{p^{(1)}, p^{(-1)}\} \longrightarrow \mathcal{E}^u_{n/m} \longrightarrow \mathcal{F}^u_{n/m} \longrightarrow 0, \tag{8.2}$$

and

$$0 \longrightarrow \mathrm{Ideal}\{q^{(1)}, q^{(-1)}\} \longrightarrow \mathcal{E}^v_{n/m} \longrightarrow \mathcal{F}^v_{n/m} \longrightarrow 0; \tag{8.3}$$

let us study the ideals and quotients separately.

Lemma 8.18. *With reference to the extension (8.1), one has*

$$\mathrm{Ideal}\{p^{(1)}, p^{(-1)}, q^{(1)}, q^{(-1)}\} \cong \bigoplus_4 \mathcal{K} \subseteq \mathcal{B}_{n/m},$$

where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space.

Similarly, with reference to the extensions (8.2) and (8.3). one has

$$\mathrm{Ideal}\{p^{(1)}, p^{(-1)}\} \cong \bigoplus_2 \mathcal{K} \subseteq \mathcal{E}^u_{n/m},$$

and

$$\mathrm{Ideal}\{q^{(1)}, q^{(-1)}\} \cong \bigoplus_2 \mathcal{K} \subseteq \mathcal{E}^v_{n/m}.$$

Proof. Let us show that $p^{(1)}$ is a minimal projection in the sense that the hereditary sub-C*-algebra generated by $p^{(1)}$ is one dimensional.

For each non-zero k, consider the element $p^{(1)}v^k p^{(1)}$. By Lemma 8.17, we have

$$p^{(1)}v^k p^{(1)} = v^k(v^{-k}p^{(1)}v^k)p^{(1)} = 0.$$

Therefore $p^{(1)}v^k p^{(1)} = 0$ in the C*-algebra $\mathcal{B}_{n/m}$. As in the proof of Lemma 5.7, we have

$$p^{(1)}f(v)p^{(1)} = (\int_{\mathbb{T}} f d\mu)p^{(1)}$$

in $\mathcal{B}_{n/m}$, where μ is Lebesgue measure on the circle.

On the other hand, by Lemma 8.17,

$$p^{(1)}(e^{2\pi i n/m} - u) = 0.$$

Hence $p^{(1)}up^{(1)} = e^{2\pi i n/m}p^{(1)}$ in $\mathcal{B}_{n/m}$. As in the proof of Theorem 5.9, we have that the hereditary sub-C*-algebra of $\mathcal{B}_{n/m}$ generated by $p^{(1)}$ is isomorphic to $\mathbb{C}$. Therefore, the ideal generated by $p^{(1)}$ is isomorphic to $\mathcal{K}$.

A similar argument shows that the hereditary sub-C*-algebra of $\mathcal{B}_{n/m}$ generated by $p^{(-1)}$ is isomorphic to $\mathbb{C}$.

Let us show that $p^{(1)}ap^{(-1)} = 0$ for any $a \in \mathcal{B}_{n/m}$. Consider the element $p^{(1)}ap^{(-1)}a^*p^{(1)}$. As in the proof of Theorem 5.9 (or of Lemma 3.3), this element can be approximated arbitrarily closely by elements $p^{(1)}ap^{(-1)}bp^{(1)}$ with b in the sub-C*-algebra generated by $\tilde{\Omega}^u_{n/m}$ and v. However, since $p^{(1)}$ is orthogonal to $v^k p^{(-1)}v^{-k}$ in $\mathcal{B}_{n/m}$ for any k by Lemma 8.17, the element $p^{(1)}ap^{(-1)}bp^{(1)}$ is zero. Hence $p^{(1)}ap^{(-1)} = 0$ for any $a \in \mathcal{B}_{n/m}$.

Therefore, the hereditary sub-C*-algebra generated by $p^{(1)}$ and $p^{(-1)}$ is $\mathbb{C}p^{(1)} + \mathbb{C}p^{(-1)}$.

The same argument applied to the projections $q^{(1)}$ and $q^{(-1)}$ shows that they are minimal, $q^{(1)}aq^{(-1)} = 0$ for any $a \in \mathcal{B}_{n/m}$, and moreover $p^{(i)}aq^{(j)} = 0$ for any $a \in \mathcal{B}_{n/m}$ where $i, j = 1, -1$. Therefore, the hereditary sub-C*-algebra generated by $\{p^{(1)}, p^{(-1)}, q^{(1)}, q^{(-1)}\}$ is isomorphic to $\bigoplus_4 \mathbb{C}$, and hence the ideal generated by these elements is isomorphic to $\bigoplus_4 \mathcal{K}$. □

Remark 8.19. By the lemma above, the exact sequences (8.1), (8.2), and (8.3) become

$$0 \longrightarrow \bigoplus_4 \mathcal{K} \longrightarrow \mathcal{B}_{n/m} \longrightarrow \mathcal{B}'_{n/m} \longrightarrow 0, \tag{8.4}$$

$$0 \longrightarrow \bigoplus_2 \mathcal{K} \longrightarrow \mathcal{E}^u_{n/m} \longrightarrow \mathcal{F}^u_{n/m} \longrightarrow 0, \tag{8.5}$$

and

$$0 \longrightarrow \bigoplus_2 \mathcal{K} \longrightarrow \mathcal{E}^v_{n/m} \longrightarrow \mathcal{F}^v_{n/m} \longrightarrow 0, \tag{8.6}$$

Let us consider the quotient algebra $\mathcal{B}'_{n/m}$. Since

$$\begin{aligned} p_0 &= p_0(v^{m+1}p_0v^{-m-1} + v^m p_0 v^{-m} + (v^{m-1}p_0v^{-m+1})) \\ &= p^{(1)} + p_0p_m + p^{(-1)} \end{aligned}$$

in $\tilde{\Omega}^u_{n/m}$, we have that $p_0 \leq p_m$ in $\mathcal{B}'_{n/m}$. By symmetry, also $p_m \leq p_0$ in $\mathcal{B}'_{n/m}$, and therefore $p_0 = p_m$ in $\mathcal{B}'_{n/m}$. For the same reason, we have that $p_k = p_{k+m}$ in $\mathcal{B}'_{n/m}$, and thus the image of $\tilde{\Omega}^u_{n/m}$ in $\mathcal{B}'_{n/m}$ is isomorphic to

$$\bigoplus_{i=1}^{m} \mathrm{C}([\frac{i-1}{m}, \frac{i}{m}])$$

with p_k the characteristic function of $\bigcup_{i=k+1}^{k+m+1}[(i-1)/m, i/m]$, and u the function

$$(t_1, \cdots, t_m) \mapsto (e^{2\pi i t_1}, \cdots, e^{2\pi i t_m}).$$

The same description holds for the image of $\tilde{\Omega}^u_{n/m}$ in $\mathcal{B}'_{n/m}$. Then, by the universal property of $\mathcal{B}_{n/m}$, the C*-algebra $\mathcal{B}'_{n/m}$ is the universal C*-algebra generated by the following generators and relations.

Set

$$I = \bigoplus_{i=1}^{m} \mathrm{C}([\frac{i-1}{m}, \frac{i}{m}])$$

and

$$J = \bigoplus_{j=1}^{m} \mathrm{C}([\frac{j-1}{m}, \frac{j}{m}]).$$

Fix the unitary $u \in I$ with

$$u : (t_1, \cdots, t_m) \mapsto (e^{2\pi i t_1}, \cdots, e^{2\pi i t_m})$$

and the unitary $v \in J$ with

$$v : (t_1, \cdots, t_m) \mapsto (e^{2\pi i t_1}, \cdots, e^{2\pi i t_m}).$$

The sub-C*-algebras of I and J generated by u and v respectively are isomorphic to $\mathrm{C}(\mathbb{T})$ in the same way that the unit circle $\mathbb{T}$ is obtained by gluing the common endpoint of the spectra of I or J.

There is a automorphism of order m on each of I and J induced by the map

$$(t_1, t_2, \cdots, t_m) \mapsto (t_m, t_1, \cdots, t_{m-1});$$

let us denote both of these by σ. Then, the C*-algebra $\mathcal{B}'_{n/m}$ is the universal C*-algebra generated by I and J with respect to the relations

$$v^* f v = \sigma^n(f) \quad \text{for any } f \in I$$

and

$$u g u^* = \sigma^n(g) \quad \text{for any } g \in J.$$

In particular, we have

$$uvu^* = \sigma^n(v) = e^{2\pi i n/m} v.$$

Let us calculate the irreducible representations of $\mathcal{B}'_{n/m}$. Let π be an irreducible representation of $\mathcal{B}'_{n/m}$. Since the automorphism σ has order m, the unitaries u^m and v^m are in the center of $\mathcal{B}'_{n/m}$, and therefore there exist z_1 and z_2 in $\mathbb{T}$ such that

$$(\bar{z}_1 u)^m = 1 \quad \text{and} \quad (\bar{z}_2 v)^m = 1.$$

Denote by A the sub-C*-algebra of $\mathcal{B}'_{n/m}$ generated by u and v. Since they satisfy the relation $uvu^* = e^{2\pi i n/m} v$, there is a canonical map from the rotation algebra $A_{n/m}$ to A. Therefore, the image of A is a $m \times m$ matrix algebra, and π sends u to

$$z_1 \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and sends v to

$$z_2 \mathrm{diag}(1, \omega, \omega^2, \cdots, \omega^{m-1})$$

where $\omega = e^{2\pi i n/m}$.

Hence the restriction of π to the C*-algebra generated by u and v satisfies

$$\pi|_{\mathrm{C}^*(u)}(f) = \sum_{k=1}^{m} f(e^{2\pi i k/m} z_1) p_k \quad \text{for any } f \in \mathrm{C}^*(u)$$

and

$$\pi|_{\mathrm{C}^*(v)}(g) = \sum_{k=1}^{m} g(e^{2\pi i k/m} z_2) q_k \quad \text{for any } g \in \mathrm{C}^*(v),$$

where $\{p_k\}$ and $\{q_k\}$ are two families of mutually orthogonal projections. In matrix form,

$$p_k = W \mathrm{diag}(\underbrace{0, \cdots, 0, 1}_{k \text{ copies}}, 0, \cdots, 0) W^*$$

and

$$q_k = \mathrm{diag}(\underbrace{0, \cdots, 0, 1}_{k \text{ copies}}, 0, \cdots, 0),$$

where W is the matrix with

$$(W)_{r,s} = \frac{1}{\sqrt{m}}(\omega^{rs}).$$

Let us call the homomorphisms in this form point evaluation maps.

Consider the restriction of π to I. Then the kernel $\ker(\pi|_I)$ must be $\dot{\bigcup}[(i-1)/m, i/m]$ except for finite many points, where $\dot{\bigcup}$ denotes the disjoint union. Otherwise, the image of u would be infinite dimensional, in contradiction with the statement that the image of A is finite dimensional. Thus, π sends I to a finite dimensional C*-algebra. A similar argument shows that π also sends J to a finite dimensional C*-algebra.

Note that the restriction of π to the C*-algebra generated by u is the direct sum of the point evaluation maps at $\{e^{2\pi ik/m}z_1; 1 \leq k \leq m\}$. Therefore, if $z_1 \notin \{e^{2\pi ik/m}; 1 \leq k \leq m\}$, the map $\pi|_I$ has to be the direct sum of the point evaluation maps at $\{\log(z_1 e^{2\pi ik/m}); 1 \leq k \leq m\}$, and hence is determined by z_1. By the same reason, if $z_2 \notin \{e^{2\pi ik/m}; 1 \leq k \leq m\}$, then the map $\pi|_J$ is the direct sum of the point evaluation maps at $\{\log(z_1 e^{2\pi ik/m}); 1 \leq k \leq m\}$, and hence is determined by z_2. Therefore, if $z_1, z_2 \notin \{e^{2\pi ik/m}; 1 \leq k \leq m\}$, the irreducible representation has dimension m, and is determined by its restriction to A.

If $z_1 \in \{e^{2\pi ik/m}; 1 \leq k \leq m\}$, then the restriction of π to the C*-algebra generated by u is the direct sum of the point evaluations at $\{e^{2\pi ik/m}; 1 \leq k \leq m\}$. Therefore, the restriction of π to the C*-algebra I has the form

$$\pi|_I(f) = \sum_{k=1}^{m}(f((\frac{k}{m})_l)p_1^{(k)} + f((\frac{k}{m})_r)p_2^{(k)}) \quad \text{for any } f \in I,$$

where $p_1^{(k)}$ and $p_2^{(k)}$ are subprojections of p_k with $p_1^{(k)} + p_2^{(k)} = p_k$, and $(k/m)_l$ and $(k/m)_r$ denote the endpoint k/m of the left-hand interval or the right-hand interval respectively.

For any k, take a test function f_k in I which is zero on all intervals $[(i-1)/m, i/m]$, $1 \leq i \leq m$ except for $[(k-1)/m, k/m]$, and furthermore, assume that $f_k(((k-1)/m)_r) = 0$ and $f_k((k/m)_l) = 1$. Then, we have

$$\pi(f_k) = p_1^{(k)}.$$

Note that $\sigma(f_k)$ is then such a test function for the interval $[k/m, (k+1)/m]$. Since $v^* f v = \sigma^n(f)$ for any $f \in I$, we have that

$$\pi(v^*)p_1^{(k)}\pi(v) = \pi(v^* f_k v) = \pi(\sigma^n(f_k)) = p_1^{(k+n)}.$$

A similar argument shows that

$$\pi(v^*)p_2^{(k)}\pi(v) = p_2^{(k+n)}.$$

If $z_1 \notin \{e^{2\pi ik/m}; 1 \leq k \leq m\}$, consider the projection

$$P_1 := p_1^{(1)} + \cdots + p_1^{(n)}.$$

It is clear that P_1 commutes with $\pi(v)$, and hence commutes with $\pi(J)$ (since the restriction of π to J is determined by its restriction to v). Therefore P_1 is a central projection, and hence is

0 or 1. The restriction of π to I is then the point evaluation map at the left endpoint (which corresponds to $P_1 = 1$) or the right endpoint (which corresponds $P_1 = 0$).

If $z_1 \notin \{e^{2\pi ik/m}; 1 \leq k \leq m\}$ and $z_2 \in \{e^{2\pi ik/m}; 1 \leq k \leq m\}$, then the situation is the same to the one above.

If both z_1 and z_2 belong to $\{e^{2\pi ik/m}; 1 \leq k \leq m\}$, then we have

$$\pi|_I(f) = \sum_{k=1}^{m}(f((\frac{k}{m})_l)p_1^{(k)} + f((\frac{k}{m})_r)p_2^{(k)}) \quad \text{for any } f \in I,$$

and

$$\pi|_J(g) = \sum_{k=1}^{m}(g((\frac{k}{m})_l)q_1^{(k)} + g((\frac{k}{m})_r)q_2^{(k)}) \quad \text{for any } g \in J,$$

where $p_1^{(k)} + p_2^{(k)} = p_k$ and $q_1^{(k)} + q_1^{(k)} = q_k$. Moreover,

$$\pi(v^*)p_1^{(k)}\pi(v) = p_1^{(k+n)} \quad \text{and} \quad \pi(v^*)p_2^{(k)}\pi(v) = p_2^{(k+n)},$$

and

$$\pi(u)q_1^{(k)}\pi(u^*) = q_1^{(k+n)} \quad \text{and} \quad \pi(u)q_2^{(k)}\pi(u^*) = q_2^{(k+n)}.$$

Thus the image of $\mathcal{B}'_{n/m}$ is generated by the operators

$$\begin{pmatrix} p_1^{(0)} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} q_1^{(0)} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

together with the $m \times m$ matrix algebras (which is generated by the images of u and v). Hence the C*-algebra generated by them is a simple quotient of $\mathrm{M}_m(\mathbb{C}) \otimes S_2$, where

$$(8.7)\quad S_2 = \{f \in \mathrm{M}_2(\mathrm{C}([0,1]));\ f(0) = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{pmatrix}, f(1) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \mu_1 \end{pmatrix} \text{ for some } \lambda_0, \mu_0, \lambda_1, \mu_1\}$$

is the universal unital C*-algebra generated by two projections. Therefore, the irreducible representations of $\mathcal{B}'_{n/m}$ can be parameterized by a square together with a splitting interval, in the following way.

Define

$$V_0 = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and

$$U_0 = \mathrm{diag}(1, \omega, \omega^2, \cdots, \omega^{m-1}).$$

Then, for any $(x, y) \in [0, 1] \times [0, 1]$, the map

$$\begin{aligned} \pi_{(x,y)}(f) &= \sum_{i=1}^{m} f(\frac{i + x - 1}{m}) e_{i,i}, \quad f \in I, \\ \pi_{(x,y)}(g) &= W(\sum_{j=1}^{m} g(\frac{j + y - 1}{m}) e_{i,i}) W^*, \quad g \in J, \end{aligned}$$

define an m-dimensional irreducible representation of $\mathcal{B}'_{n/m}$, where 0 and $1/m$ are considered to be the points in the first interval.

For any $t \in [0, 1]$, define the following map with range $\mathrm{M}_m(\mathbb{C}) \otimes \mathrm{M}_2(\mathbb{C})$:

$$\begin{aligned} \pi_t(f) &= \sum_{i=1}^{m} (f(\frac{i}{m})(e_{i,i} \otimes r_t) + f(\frac{i-1}{m})(e_{i-1,i-1} \otimes (1 - r_t))), \\ \pi_t(g) &= W(\sum_{i=1}^{m} (g(\frac{i}{m})(e_{i,i} \otimes s_t) + g(\frac{i-1}{m})(e_{i-1,i-1} \otimes (1 - s_t)))) W^*, \end{aligned}$$

where

$$r_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad s_t = \begin{pmatrix} 1 - t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & t \end{pmatrix}.$$

This gives a $2m$-dimensional irreducible representation of $\mathcal{B}'_{n/m}$ if $t \neq 0, 1$. If $t = 0$, the map is unitarily equivalent to the direct sum of the two m-dimensional irreducible representations which correspond to the vertices $(0, 0)$ and $(1, 1)$ of the square referred to above. If $t = 1$, the map is then unitarily equivalent to the direct sum of the two m-dimensional irreducible representations corresponding to the vertices $(1, 0)$ and $(0, 1)$ of the square.

Using the description of the irreducible representations above, let us set $\mathcal{B}'_{n/m}$ with a concrete C*-algebra. As defined in [BEEK92], let us set

$$W_1 = U_0^{n'} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \rho^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho^{-(m-1)} \end{pmatrix} \tag{8.8}$$

where $\rho = e^{2\pi i \frac{1}{m}}$, and $nn' = -1 \mod m$ (and $0 < n' < m$), and define

$$W_2 = V_0^{n''} = \begin{pmatrix} 0 & \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix} \\ \underbrace{\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix}}_{n''} & 0 \end{pmatrix} \tag{8.9}$$

where $nn'' = 1 \mod m$ (and $0 < n'' < m$). Set $\alpha_1 = \mathrm{Ad}W_1$ and $\alpha_2 = \mathrm{Ad}W_2$. Then the rational rotation algebra $A_{n/m}$ can be described as the algebra of functions f from $[0,1]\times[0,1]$ to $\mathrm{M}_m(\mathbb{C})$ such that

$$\begin{aligned} f(x,1) &= \alpha_1(f(x,0)), \quad 0 \le x \le 1, \\ f(1,y) &= \alpha_2(f(0,y)), \quad 0 \le y \le 1. \end{aligned}$$

Note that the algebra S_2 has four one-dimensional irreducible representations, which correspond to λ_0, μ_0, λ_1, and μ_1 of (8.7). For any $b \in S_2$, let us denote by $b(0,0)$, $b(0,1)$, $b(1,0)$, and $b(1,1)$ the images of b under these irreducible representations, respectively.

Consider the following concrete C*-algebra: Set $E := \mathrm{M}_m(\mathrm{C}([0,1]\times[0,1])) \oplus (\mathrm{M}_m \otimes S_2)$, and define

$$B'_{n/m} := \left\{ (a,b) \in E; \begin{array}{ll} a(0,1) = \alpha_2^{-1} b(1,0), & a(1,1) = b(0,1), \\ a(0,0) = \alpha_1^{-1}\alpha_2^{-1} b(0,0), & a(1,0) = \alpha_1^{-1} b(1,1). \end{array} \right\},$$

and consider the elements

$$\begin{aligned} \tilde{u} &= (e^{2\pi i x/m} U_0, e^{2\pi i 1/m} U_0 \otimes 1), \\ \tilde{v} &= (e^{2\pi i y/m} V_0, e^{2\pi i 1/m} V_0 \otimes 1), \\ \tilde{p}^{(1)} &= (e_{1,1}, e_{1,1} \otimes r_t + e_{m,m} \otimes (1 - r_t)), \\ \tilde{q}^{(1)} &= (W e_{1,1} W^*, W(e_{1,1} \otimes s_t + e_{m,m} \otimes (1 - s_t)) W^*). \end{aligned}$$

Then the elements $\tilde{u}$ and $\tilde{p}^{(1)}$ generate the interval algebra I, the elements $\tilde{v}$ and $\tilde{q}^{(1)}$ generate the interval algebra J, and I and J generate the C*-algebra $B'_{n/m}$. Moreover, the irreducible representations of the C*-algebra $B'_{n/m}$ exhaust all the irreducible representations of the universal C*-algebra $\mathcal{B}'_{n/m}$. Therefore, we get the following lemma.

Lemma 8.20. *The universal C*-algebra $\mathcal{B}'_{n/m}$ is isomorphic to the concrete C*-algebra $B'_{n/m}$. In particular, the C*-algebra $\mathcal{B}'_{n/m}$ is a subhomogeneous C*-algebra.*

Remark 8.21. By the exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{4} \mathcal{K} \longrightarrow \mathcal{B}_{n/m} \longrightarrow \mathcal{B}'_{n/m} \longrightarrow 0,$$

the C*-algebra $\mathcal{B}_{n/m}$ is of type I.

Observe that by shrinking the square to a single point, the C*-algebra $\mathcal{B}'_{n/m}$ is homotopic to the C*-algebra $\mathrm{M}_m(\mathbb{C}) \otimes D$, where D is the dimension drop circle algebra

$$D := \{f \in \mathrm{M}_2(\mathrm{C}(\mathbb{T}));\ f(1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \text{ for some } \lambda \in \mathbb{C}\}. \tag{8.10}$$

A direct calculation shows that $\mathrm{K}_0(\mathcal{B}'_{n/m}) \cong \mathrm{K}_0(D) \cong \mathbb{Z}$ with $[1] = m$, and $\mathrm{K}_1(\mathcal{B}'_{n/m}) \cong \mathrm{K}_1(D) \cong \mathbb{Z}$, and the six-term exact sequence associated with the extension (8.4) becomes

$$0 \longrightarrow \mathrm{K}_1(\mathcal{B}_{n/m}) \longrightarrow \mathbb{Z} \xrightarrow{\mathrm{Ind}} \bigoplus_4 \mathbb{Z} \xrightarrow{i} \mathrm{K}_0(\mathcal{B}_{n/m}) \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0\ ,$$

where Ind is the index map. Since $\mathbb{Z}$ is projective, there is a map $j : \mathbb{Z} \to \mathrm{K}_0(\mathcal{B}_{n/m})$ such that $\pi \circ j = \mathrm{id}$. Therefore, we have the following lemma.

Lemma 8.22. *For each rational number r, the map $(\mathrm{id} - j \circ \pi, \pi)$ induces an isomorphism*

$$K_0(\mathcal{B}_r) \cong (\bigoplus_4 \mathbb{Z}/\mathrm{Ind}(\mathbb{Z})) \oplus \mathbb{Z}.$$

Let us consider the C*-algebras $\mathcal{F}^u_{n/m}$ and $\mathcal{F}^v_{n/m}$. The C*-algebra $\mathcal{F}^u_{n/m}$ is the universal C*-algebra generated by the following generators and relations.

Set

$$I = \bigoplus_{i=1}^{m} \mathrm{C}([\frac{i-1}{m}, \frac{i}{m}]).$$

Fix the unitary $u \in I$ with

$$u : (t_1, \cdots, t_m) \mapsto (e^{2\pi i t_1}, \cdots, e^{2\pi i t_m}).$$

There is a automorphism of order m of I induced by the map

$$(t_1, t_2, \cdots, t_m) \mapsto (t_m, t_1, \cdots, t_{m-1}),$$

and we denote both of them by σ. Then, the C*-algebra $\mathcal{F}^u_{n/m}$ is the universal C*-algebra generated by I and and a unitary v with respect to the relations

$$v^* f v = \sigma^n(f) \quad \text{for any } f \in I.$$

Then, it is clear that the C*-algebra $\mathcal{F}^u_{n/m}$ is the sub-C*-algebra of $\mathcal{B}'_{n/m}$ generated by the corresponding elements. Therefore, by Lemma 8.20, the C*-algebra $\mathcal{F}^u_{n/m}$ is isomorphic to

$$F^u_{n/m} = \mathrm{C}^*\{\tilde{u}, \tilde{v}, \tilde{p}^{(1)}\} \subseteq B'_{n/m}.$$

Since the restrictions of irreducible representations of $B'_{n/m}$ to $F^u_{n/m}$ are constant on the open subinterval $(0, 1)$ of the spectra of $\mathrm{M}_m(\mathbb{C}) \otimes S_2$, one has

$$F^u_{n/m} = \mathrm{C}^*\{e^{2\pi i x/m} U_0, e^{2\pi i y/m} V_0, e_{1,1}\} \subseteq \mathrm{C}([0, 1]^2, \mathrm{M}_m),$$

and therefore,

$$F^u_{n/m} = \{f \in \mathrm{C}([0, 1]^2, \mathrm{M}_m);\ f(x, 1) = \mathrm{Ad}W_1(f(x, 0))\},$$

where the unitary W_1 is defined by (8.8).

Since the unitary group of any matrix algebra is path connected, the C*-algebra $F^u_{n/m}$ is isomorphic to the homogeneous C*-algebra with spectra the cylinder $\mathbb{T}\times[0,1]$, that is,

$$F^u_{n/m}\cong \mathrm{C}(\mathbb{T}\times[0,1],\mathrm{M}_m(\mathbb{C})).$$

Hence, $\mathrm{K}_0(F^u_{n/m})=\mathbb{Z}$ and $\mathrm{K}_1(F^u_{n/m})=\mathbb{Z}$. From the identification above, it is clear that v represents a generator of $\mathrm{K}_1(F^u_{n/m})$. In particular, it lifts to a unitary in $\mathcal{E}^u_{n/m}$ in the extension (8.5), and therefore, the index map associated with (8.5) is zero.

The C*-algebra $\mathcal{F}^v_{n/m}$ is defined in a similar way, and the argument above also valids. So we have the following lemma.

Lemma 8.23. *The universal C*-algebra $\mathcal{F}^u_{n/m}$ is isomorphic to the homogeneous C*-algebra $\mathrm{C}(\mathbb{T}\times[0,1],M_m)$. Moreover, the index map associated with (8.5) is zero. The same statement holds for $\mathcal{F}^v_{n/m}$.*

Corollary 8.24. *The K-theory of the C*-algebras $\mathcal{E}^u_{n/m}$ and $\mathcal{E}^v_{n/m}$ is given by*

$$K_0(\mathcal{E}^u_{n/m})=(\mathbb{Z}\oplus\mathbb{Z})\oplus\mathbb{Z},\quad K_1(\mathcal{E}^u_{n/m})=\mathbb{Z}$$

and

$$K_0(\mathcal{E}^v_{n/m})=(\mathbb{Z}\oplus\mathbb{Z})\oplus\mathbb{Z},\quad K_1(\mathcal{E}^v_{n/m})=\mathbb{Z}$$

Proof. By Lemma 8.23, the six-term exact sequence associated to the extension (8.5) becomes

$$0\longrightarrow \mathrm{K}_1(\mathcal{E}^u_{n/m})\longrightarrow\mathbb{Z}\xrightarrow{\mathrm{Ind}}\mathbb{Z}\oplus\mathbb{Z}\xrightarrow{i}\mathrm{K}_0(\mathcal{E}^u_{n/m})\xrightarrow{\pi}\mathbb{Z}\longrightarrow 0,$$

and the index map is zero. Therefore, $\mathrm{K}_0(\mathcal{E}^u_{n/m})=(\mathbb{Z}\oplus\mathbb{Z})\oplus\mathbb{Z}$ and $\mathrm{K}_1(\mathcal{E}^u_{n/m})=\mathbb{Z}$. The same argument applies to $\mathcal{E}^v_{n/m}$. ☐

Corollary 8.25. *The index map associated to the extension (8.4) is zero.*

Proof. Since $\mathcal{B}_{n/m}\cong\mathcal{E}^u_{n/m}*_{A_{n/m}}\mathcal{E}^v_{n/m}$ (see Remark 8.15), we have

$$\begin{array}{ccccc}
\mathrm{K}_0(A_{n/m}) & \xrightarrow{(\iota_0^0,\iota_1^0)} & \mathrm{K}_0(\mathcal{E}^u_{n/m})\oplus\mathrm{K}_0(\mathcal{E}^v_{n/m}) & \longrightarrow & \mathrm{K}_0(\mathcal{B}_{n/m})\\
\uparrow & & & & \downarrow\\
\mathrm{K}_1(\mathcal{B}_{n/m}) & \longleftarrow & \mathrm{K}_1(\mathcal{E}^u_{n/m})\oplus\mathrm{K}_1(\mathcal{E}^v_{n/m}) & \xleftarrow[(\iota_0^1,\iota_1^1)]{} & \mathrm{K}_1(A_{n/m}),
\end{array}$$

where ι_0 and ι_1 are the embeddings of $A_{n/m}$ into $\mathcal{E}^u_{n/m}$ and $\mathcal{E}^v_{n/m}$ respectively, and ι_0^i and ι_1^i denote the maps induced on the K_i-groups. Since $(\iota_0^1,\iota_1^1)=\mathrm{id}$, $\iota_0^0(1,0)=(0,0,0)$, and $\iota_0^0(0,1)=(0,0,1)$, and $\iota_1^0(1,0)=(0,0,0)$ and $\iota_1^0(0,1)=(0,0,1)$ (identify $\mathrm{K}_0(A_{n/m})$ with $\mathbb{Z}\bigoplus\mathbb{Z}$ with the first copy of $\mathbb{Z}$ representing the infinitesimal elements), we have

$$0\longrightarrow\mathrm{K}_1(\mathcal{B}_{n/m})\longrightarrow\mathbb{Z}\oplus\mathbb{Z}\xrightarrow{(\iota_0^0,\iota_1^0)}(\bigoplus{}_3\mathbb{Z})\oplus(\bigoplus{}_3\mathbb{Z})\longrightarrow\mathrm{K}_0(\mathcal{B}_{n/m})\longrightarrow 0\ ,$$

and therefore $\mathrm{K}_1(\mathcal{B}_{n/m})=\mathbb{Z}$ and $\mathrm{K}_0(\mathcal{B}_{n/m})=\bigoplus_5\mathbb{Z}$. Comparing to Lemma 8.22, one has that the index map must be zero, as desired. ☐

It is interesting to point out that only with the upper semicontinuity of the field and Lemma 8.22 (without knowing whether the index map is zero), one can obtain certain K-theory informations on irrational fibres.

Theorem 8.26. *There is a dense G_δ subset V of irrational numbers in the interval $[0,1)$ such that for any $\theta \in V$,*

$$(K_0(\mathcal{B}_\theta), K_0^+(\mathcal{B}_\theta), [\mathbf{1}_{\mathcal{B}_\theta}]) \cong (\mathbb{Z}+\theta\mathbb{Z}, (\mathbb{Z}+\theta\mathbb{Z})\cap\mathbb{R}^+, 1).$$

The isomorphism is induced by the canonical tracial state of $\mathcal{B}_\theta$.

Proof. Fix an increasing sequence of finite subsets $(\mathcal{F}_n)$ of $\mathcal{B}$ with dense union, and let us consider the restriction of the field $(\mathcal{B}_\theta)$ to a closed interval $[\varepsilon, 1-\varepsilon]$.

For each $\mathcal{F}_n$ and each natural number k, since the field is upper semicontinuous, there is a $\delta_{n,k} > 0$ such that for any $c \in \mathrm{M}_k(\mathcal{F}_n)$ and any $r, s \in [\varepsilon, 1-\varepsilon]$ with $|r-s| \leq \delta_{n,k}$, if $\pi_r(c)$ is close to a projection, within distance $1/32$, then $\pi_s(c)$ is close to a projection within distance $1/16$.

Fix a finite subset $\mathcal{F}_n$ and a natural number k. For each rational r, set

$$\mathcal{P}'_{r,n,k} = \{f \in \mathrm{M}_k(\mathcal{F}_n);\ \|\pi_r(f) - p\| \leq 1/16 \text{ for some projection } p\}.$$

Since $\mathrm{K}_0(\mathcal{B}_r)$ is finitely generated (as an abelian group), we can fix a finite set of projections $\mathcal{P}_r$ in $\mathrm{M}_{k'}(\mathcal{B}_r)$ which generates $\mathrm{K}_0(\mathcal{B}_r)$. We may assume that $k' \geq k$, and that for each $f \in \mathcal{P}'_{r,n,k}$, there is a $p \in \mathcal{P}_r$ such that $\|p - \pi_r(f)\| \leq 1/16$, and $\mathcal{P}_r$ contains the minimal projections $\{p^{(1)}, p^{(-1)}, q^{(1)}, q^{(-1)}\}$ (as indicated in the proof of Lemma 8.18) which generate the ideal $\bigoplus_4 \mathcal{K}$.

Since $(\mathcal{B}_\theta)$ is upper semicontinuous at r, each projection in $\mathcal{P}_r$ can be extended approximately to a neighbourhood, and thus there is a neighbourhood $V(\mathcal{F}_n, k, r)$ such that for each $s \in V(\mathcal{F}_n, k, r)$, there is a homomorphism $\psi_{r,s}$ from $\mathrm{K}_0(\mathcal{B}_r)$ to $\mathrm{K}_0(\mathcal{B}_s)$ (as abelian groups).

Note that if $f \in \mathcal{P}'_{r,n,k}$ and if p is a projection in $\mathcal{P}_r$ with $\|p - \pi_r(f)\| \leq 1/16$, we may assume that $V(\mathcal{F}_n, k, r)$ is small enough such that there is a projection $q \in \mathcal{B}_s$ with $\psi_{r,s}([p]) = [q]$ and $\|q - \pi_s(f)\| \leq 1/8$ for each s. Furthermore, let us assume that the length of $V(\mathcal{F}_n, k, r)$ is less than $\delta_{n,k}$.

Set

$$V(\mathcal{F}_n, k) = \bigcup_r V(\mathcal{F}_n, k, r).$$

This is a dense open subset of $[\varepsilon, 1-\varepsilon]$. Therefore the set

$$V_\varepsilon = \bigcap_{n,k} V(\mathcal{F}_n, k)$$

is non-empty.

Let $\varepsilon \to 0$, and set $V = \bigcup_\varepsilon V_\varepsilon$. Then V is a dense G_δ subset of $[0,1)$. For each $\theta \in V$, let us show that $\mathrm{K}_0(\mathcal{B}_\theta)$ is an inductive limit of quotients of $\{\mathrm{K}_0(\mathcal{B}_r);\ r \in \mathbb{Q}\}$ as abelian groups, and using this to show that

$$(\Phi(\mathrm{K}_0(\mathcal{B}_\theta)), \Phi(\mathrm{K}_0^+(\mathcal{B}_\theta)), \Phi([\mathbf{1}_{\mathcal{B}_\theta}])) = (\mathbb{Z}+\theta\mathbb{Z}, (\mathbb{Z}+\theta\mathbb{Z})\cap\mathbb{R}^+, 1)$$

with Φ inducing an isomorphism of ordered groups.

For any finite set of projections $\{q_1, \cdots, q_m\}$ in $\mathrm{M}_k(\mathcal{B}_\theta)$, we have to show that there is a rational number r and a homomorphism $\psi_{r,\theta}$ from $\mathrm{K}_0(\mathcal{B}_r)$ to $\mathrm{K}_0(\mathcal{B}_\theta)$ such that $\{[q_1], \cdots, [q_n]\}$ is in the image of $\psi_{r,\theta}$.

Pick $\mathcal{F}_n$ such that each q_i is close to $\pi_\theta(f_i)$ for some $f_i \in \mathrm{M}_k(\mathcal{F}_n)$, with distance $1/32$. Since $\theta \in V_\varepsilon$, we have that $\theta \in V(\mathcal{F}_n, k)$, and therefore

$$\theta \in V(\mathcal{F}_n, k, r)$$

for some rational number r. Since the length of $V(\mathcal{F}_n, k, r)$ is less than $\delta_{n,k}$, there is a projection p_i in $\mathcal{B}_r$ such that p_i is close to $\pi_r(f_i)$, within distance $1/16$. In other words, $f_i \in \mathcal{P}'_{r,n,k}$. Therefore, there is a projection q'_i in $\mathcal{B}_\theta$ with $[q'_i]$ is in the image of $\psi_{r,\theta}$ and $\|q'_i - \pi_\theta(f_i)\| \leq 1/8$. In particular, $\|q_i - q'_i\| \leq 1/4$. Hence, we have that $[q_i] = [q'_i]$ and $[q_i]$ is in the image of $\psi_{r,\theta}$.

Therefore, $\mathrm{K}_0(\mathcal{B}_\theta)$ is an inductive limit of quotients of $\{\mathrm{K}_0(\mathcal{B}_r);\ r \in \mathbb{Q}\}$. Let us study the map $\psi_{r,\theta}$ more closely. By Lemma 8.22,

$$\mathrm{K}_0(\mathcal{B}_r) \cong (\bigoplus_4 \mathbb{Z}/\mathrm{Ind}(\mathbb{Z})) \oplus \mathbb{Z}$$

where the direct summand $\bigoplus_4 \mathbb{Z}/\mathrm{Ind}(\mathbb{Z})$ comes from the ideal $\bigoplus_4 \mathcal{K}$, which is generated by the minimal projections $\{p^{(1)}, p^{(-1)}, q^{(1)}, q^{(-1)}\}$.

Let us assume that $r = n/m$ with m and n relatively prime, and consider the minimal projection $p^{(1)} \in \mathcal{B}_r$. The minimal projection comes from the section $(p_\theta v_\theta^{m+1} p_\theta v_\theta^{-m-1})$ which lies inside (Ω^u_θ). In particular, for any irrational θ_0, the image $\pi_{\theta_0}(p_\theta v_\theta^{m+1} p_\theta v_\theta^{-m-1})$ is a projection in the commutative sub-C*-algebra $\Omega^u_{\theta_0}$ of $\mathcal{B}_{\theta_0}$ with trace $(\theta_0 - \{(m+1)\theta_0\})$ where $\{x\}$ denotes $x' \in [0, 1]$ with $x - x' \in \mathbb{Z}$. Similar calculations show that the minimal projection $q^{(1)}$ can be extended to a projection in the commutative sub-C*-algebra $\Omega^v_{\theta_0}$ of $\mathcal{B}_{\theta_0}$ with the same trace. Since any projection in $\Omega^u_{\theta_0}$ (or $\Omega^u_{\theta_0}$) is unitarily equivalent to the projection in A_{θ_0} with the same trace, we have that $\pi_{r,\theta}([p^{(1)}]) = \pi_{r,\theta}([q^{(1)}])$.

Same argument shows that $\pi_{r,\theta}([p^{(-1)}]) = \pi_{r,\theta}([q^{(-1)}])$. Moreover, from the construction of these minimal projections, we have that the images of $p^{(1)}$ and $q^{(1)}$ are non-zero if and only if the images of $p^{(-1)}$ and $q^{(-1)}$ are zero. Therefore, the image of the direct summand $\bigoplus_4 \mathbb{Z}/\mathrm{Ind}(\mathbb{Z})$ is $\mathbb{Z}$, and hence the image of $\psi_{r,\theta}$ is a quotient of $\mathbb{Z} \oplus \mathbb{Z}$.

Let us show that the image is just $\mathbb{Z} \oplus \mathbb{Z}$. Suppose this were not true, then $\mathrm{K}_0(\mathcal{B}_\theta)$ is an inductive limit of proper quotients of $\mathbb{Z} \oplus \mathbb{Z}$. Consider the state

$$\Phi : \mathrm{K}_0(\mathcal{B}_\theta) \to \mathbb{R}$$

induced by the canonical tracial state. Then the subgroup $\Phi(\mathrm{K}_0(\mathcal{B}_\theta))$ must be an inductive limit of copies $\mathbb{Z}$, which contradicts the fact that $\mathbb{Z} + \theta\mathbb{Z}$ is a subgroup of $\Phi(\mathrm{K}_0(\mathcal{B}_\theta))$.

Therefore, $\mathrm{K}_0(\mathcal{B}_\theta)$ is an inductive limit of $\mathbb{Z} \oplus \mathbb{Z}$ in the category of countable abelian groups with the connection map being injective.

Fix a subgroup $\mathbb{Z} \bigoplus \mathbb{Z}$ of $\mathrm{K}_0(\mathcal{B}_\theta)$ which is the image of $\psi_{r,\theta}$, and consider the restriction of the canonical tracial state Φ to it. Let us assume that $\psi_{r,\theta}$ contains $[p]$ and $[\mathbf{1}_{\mathcal{B}_\theta}]$ where p is the spectral projection of u corresponding $[0, \theta)$ (in other words, we assume that $\mathcal{F}_n$ contains the sections (p_θ) and $(\mathbf{1})$). Let us show that $\Phi(\mathbb{Z} \oplus \mathbb{Z}) \subseteq \mathbb{Z} + \theta\mathbb{Z}$.

Note that the K_0-class of each minimal projection of $\{p^{(1)}, p^{(-1)}, q^{(1)}, q^{(-1)}\}$ in $\mathcal{B}_r$ is sent to the element $(1,0) \in \mathbb{Z}\oplus\mathbb{Z} \subseteq \mathrm{K}_0(\mathcal{B}_\theta)$, which has trace $\theta - \{(m+1)\theta\}$ (recall that x denotes the largest integer less than or equal to x), or sent to zero (depending on whether r sits on the left-hand side or right-hand side of θ). We then have

$$\Phi((1,0)) = k_1 + k_2\theta$$

for integers k_1 and k_2 with $k_2 \neq 0$.

Note that when we identify $\psi_{r,\theta}(\mathrm{K}_0(\mathcal{B}_r))$ with $\mathbb{Z}\oplus\mathbb{Z}$, we use the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \psi_{r,\theta}(\mathrm{K}_0(\mathcal{B}_r)) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

with $\pi(\psi_{r,\theta}([\mathbf{1}_{\mathcal{B}_r}])) = m$ and $\pi(\psi_{r,\theta}([p])) = n$, where π denotes the quotient map. Then the element $\psi_{r,\theta}([\mathbf{1}_{\mathcal{B}_r}])$ (which is the same as $[\mathbf{1}_{\mathcal{B}_\theta}]$) has the form (m', m) for some integer m', and the element $[p]$ has the form (n', n) for some integer n'. (Recall that p comes from the section (p_θ), and the K_0-class of the restriction of (p_θ) to r has the form (k, n) for some $k \in (\bigoplus_4 \mathbb{Z})/\mathrm{Ind}(\mathbb{Z})$. Therefore, $[p]$ has the form $\psi_{r,\theta}((k,n)) = (n', n)$ for some integer n'.)

Set $s = \Phi((0,1)) \in \mathbb{R}$. If $p^{(1)}$ is non-zero, then $p^{(1)}$ has trace $\theta - \{(m+1)\theta\} = -m\theta + n$. Since $\Phi(\psi_{r,\theta}([1_{\mathcal{B}_r}])) = \Phi(m', m) = 1$, we have

$$1 = m'(\theta - \{(m+1)\theta\}) + ms.$$

That is

$$1 = m'(-m\theta + n) + ms = ms - m'm\theta + m'n.$$

We then have

$$s = \frac{1 - m'n}{m} + m'\theta.$$

In other words, $s = s_1 + m'\theta$ for a rational number s_1.

Let us carry out a similar calculation for $[p]$. Note that $\Phi(\psi_{r,\theta}([p])) = \theta$. We have that $\Phi((n', n)) = \theta$, and then

$$\theta = n'(-m\theta + n) + ns = n'(-m\theta + n) + n(s_1 + m'\theta).$$

Therefore,

$$s_1 = -n',$$

and we have

$$s = -n' + m'\theta \in \mathbb{Z} + \theta\mathbb{Z}.$$

(Note that from the calculation above, we have that $n/m < \theta < n'/m'$ with $m'n - mn' = 1$.)

If $p^{(1)}$ is zero, then $p^{(-1)}$ must be non-zero, and then an argument similar to the one above also shows that $s \in \mathbb{Z} + \theta\mathbb{Z}$.

Since the image $\Phi(\mathbb{Z}\oplus\mathbb{Z})$ is the subgroup of $\mathbb{R}$ generated by $\Phi((0,1)) = s$ and $\Phi((1,0)) = k_1 + k_2\theta$, we have

$$\Phi(\mathbb{Z}\oplus\mathbb{Z}) \subseteq \mathbb{Z} + \theta\mathbb{Z}.$$

Moreover, the restriction of Φ to $\mathbb{Z}\oplus\mathbb{Z}$ is injective.

Since $\mathrm{K}_0(\mathcal{B}_\theta)$ is the inductive limit of the subgroups above, we have

$$\Phi(\mathrm{K}_0(\mathcal{B}_\theta)) \subseteq \mathbb{Z} + \theta\mathbb{Z},$$

and the map Φ is injective. On the other hand, it is clear that

$$\Phi(\mathrm{K}_0(\mathcal{B}_\theta)) \supseteq \mathbb{Z} + \theta\mathbb{Z}$$

with

$$\Phi(\mathrm{K}_0^+(\mathcal{B}_\theta)) \supseteq (\mathbb{Z} + \theta\mathbb{Z}) \cap \mathbb{R}^+.$$

Then we have

$$(\mathrm{K}_0(\mathcal{B}_\theta), \mathrm{K}_0^+(\mathcal{B}_\theta), [\mathbf{1}_{\mathcal{B}_\theta}]) \cong (\mathbb{Z} + \theta\mathbb{Z}, (\mathbb{Z} + \theta\mathbb{Z}) \cap \mathbb{R}^+, 1),$$

with the isomorphism induced by Φ, as desired. □

Remark 8.27. The method used in the proof was also used by A. Dean in [Dea01] to show that certain C*-algebras are inductive limits of semiprojective type I C*-algebras. In the following part of the paper, we shall use this method again to obtain the AF structure of irrational extended rotation algebras.

Remark 8.28. It was pointed out to us by Hanfeng Li that if one regards the irrational extended rotation algebra as the amalgamated free product of two Cuntz–Putnam algebras over the usual rotation algebra, it follows from the Cuntz–Germain–Thomsen exact sequence [Tho03] that

$$(\mathrm{K}_0(\mathcal{B}_\theta), \mathrm{K}_0^+(\mathcal{B}_\theta), [\mathbf{1}_{\mathcal{B}_\theta}]) \cong (\mathbb{Z} + \theta\mathbb{Z}, (\mathbb{Z} + \theta\mathbb{Z}) \cap \mathbb{R}^+, 1)$$

and $\mathrm{K}_1(\mathcal{B}_\theta) \cong \{0\}$ for any irrational θ. (We include our original proof of Theorem 8.26 above as an illustration in a simpler context of the method used in the proof of Theorem 8.39 below.)

Note that this method also works for general (irrational) extended rotation algebras.

8.7. **AF structure of (certain) extended rotation algebras.** Recall that the standard quotient of the C*-algebra $\mathcal{B}_{n/m}$ is given by

$$B'_{n/m} := \left\{ (a,b) \in E; \begin{array}{ll} a(0,1) = \alpha_2^{-1} b(1,0), & a(1,1) = b(0,1), \\ a(0,0) = \alpha_1^{-1}\alpha_2^{-1} b(0,0), & a(1,0) = \alpha_1^{-1} b(1,1). \end{array} \right\},$$

where $E := \mathrm{C}([0,1]^2, \mathrm{M}_m) \oplus (\mathrm{M}_m \otimes S_2)$. Consider the elements

$$\begin{array}{rcl} \tilde{u} &=& (e^{2\pi i x/m} U_0, e^{2\pi i 1/m} U_0 \otimes 1), \\ \tilde{v} &=& (e^{2\pi i y/m} V_0, e^{2\pi i 1/m} V_0 \otimes 1), \\ \tilde{p}_e &=& (e_{1,1}, e_{1,1} \otimes r_t + e_{m,m} \otimes (1 - r_t)), \\ \tilde{q}_e &=& (W e_{1,1} W^*, W(e_{1,1} \otimes s_t + e_{m,m} \otimes (1 - s_t)) W^*). \end{array}$$

Then, the images of the standard generators $\{u, p, v, q\}$ of $\mathcal{B}_{n/m}$ in $B'_{n/m}$ are

$$\tilde{u},\ \tilde{v},\ \sum_{i=0}^{n-1} \tilde{v}^{ik} \tilde{p}_e \tilde{v}^{-ik},\ \text{and}\ \sum_{i=0}^{n-1} \tilde{u}^{ik} \tilde{q}_e \tilde{u}^{-ik},$$

respectively, where $kn = 1 \mod m$.

Let us consider the sub-C*-algebra

$$D_{n/m} := \{(a,b) \in E; a \text{ is constant}\} \subseteq B'_{n/m},$$

and consider

$$\mathcal{C}_{n/m} := \pi^{-1}(D_{n/m}) \subseteq \mathcal{B}_{n/m}.$$

We then have the following lemma.

Lemma 8.29. *The distance from each of the standard generators u, p, v, and q to the sub-C*-algebra $\mathcal{C}_{n/m}$ is strictly less than $2\pi/m$.*

Proof. Since $\mathcal{C}_{n/m}$ is the pre-image of $D_{n/m}$, it is enough to show that the distance between the image of each of the generators to $D_{n/m}$ is strictly less than $2\pi/m$. It is clear that

$$\mathrm{dist}(\tilde{u}, D_{n/m}) \leq \left| e^{2\pi i \frac{1}{m}} - 1 \right| < 2\pi/m,$$

and

$$\mathrm{dist}(\tilde{v}, D_{n/m}) \leq \left| e^{2\pi i \frac{1}{m}} - 1 \right| < 2\pi/m.$$

Note that $\sum_{i=0}^{n-1} \tilde{v}^{ik}\tilde{p}_e\tilde{v}^{-ik} \in D_{n/m}$ and $\sum_{i=0}^{n-1} \tilde{u}^{ik}\tilde{q}_e\tilde{u}^{-ik} \in D_{n/m}$. One has therefore that the distance from the image of each generator to $D_{n/m}$ is strictly less than $2\pi/m$, as desired. □

Remark 8.30. It is clear that the C*-algebra $D_{n/m}$ is isomorphic to $D \otimes \mathrm{M}_m(\mathbb{C})$, where D is the dimension drop circle C*-algebra as described in (8.10). It follows from (8.4) that there is an extension

$$0 \longrightarrow \bigoplus_4 \mathcal{K} \longrightarrow \mathcal{C}_{n/m} \longrightarrow D_{n/m} \longrightarrow 0, \tag{8.11}$$

and by Corollary 8.25, the index map associated with it is zero.

We shall show that the C*-algebra $\mathcal{C}_{n/m}$ can be locally approximated by dimension drop circle algebras.

Lemma 8.31. *Let D be a dimension drop circle C*-algebra as described in (8.10). Assume that the index map associated with an extension*

$$0 \longrightarrow \bigoplus_4 \mathcal{K} \longrightarrow \mathcal{C} \longrightarrow D \otimes \mathrm{M}_m(\mathbb{C}) \longrightarrow 0$$

is zero. Then, for any finite subset $\mathcal{F} \subseteq \mathcal{C}$, there is an approximate unit (p_i) for $\bigoplus_4 \mathcal{K}$ such that each p_i is a finite rank projection, and for any $a \in \mathcal{F}$,

$$\lim_{i\to\infty} \|p_i a - a p_i\| = 0.$$

Proof. Without loss of generality, we may assume that $\mathcal{K}$ is infinite dimensional.

Let us consider the case that the extension is essential. Since the C*-algebra D satisfies the Universal Coefficient Theorem (UCT), one has

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(D), \bigoplus_4 \mathbb{Z}) \longrightarrow \mathrm{Ext}(D, \bigoplus_4 \mathcal{K}) \xrightarrow{\gamma} \mathrm{Hom}(\mathrm{K}_1(D), \bigoplus_4 \mathbb{Z}) \longrightarrow 0\ ,$$

where the map γ sends each extension to the associated index map. Since the abelian group $\mathrm{K}_0(D)$ is free and the associated index map is zero, the given extension is trivial. Thus, there is a map

$$\eta : D \otimes \mathrm{M}_m(\mathbb{C}) \to \bigoplus_4 B(\mathcal{H}) \cong \mathcal{M}(\bigoplus_4 \mathcal{K})$$

such that the C*-algebra $\mathcal{C}$ is generated by the image of D and $\bigoplus_4 \mathcal{K}$. Then, we may assume that $\mathcal{F} = \eta(\mathcal{F}')$ for some $\mathcal{F}' \subseteq D \otimes \mathrm{M}_m(\mathbb{C})$.

We assert that for any $\varepsilon > 0$, there is an approximate unit (p_i) for $\bigoplus_4 \mathcal{K}$ with each p_i a finite rank projection, such that for sufficiently large i,

$$\|[p_i, a]\| < \varepsilon, \quad a \in \mathcal{F}.$$

Indeed, since $D \otimes \mathrm{M}_m(\mathbb{C})$ is quasi-diagonal, there exist a faithful representation $\sigma : D \otimes \mathrm{M}_m(\mathbb{C}) \to \bigoplus_4 B(\mathscr{H})$ and finite rank projections $\{p_i\}$ converging to the identity strongly such that

$$\lim_{i\to\infty} \|[p_i, \pi(d)]\| = 0, \quad d \in D \otimes \mathrm{M}_m(\mathbb{C}).$$

By Voiculescu's Theorem, the representation σ is approximately unitarily equivalent to η. Thus the assertion holds.

Applying the assertion for $\varepsilon_k = 1/2^k$, we have approximate units $(p_i^{(k)};\ i \in \mathbb{N})$ such that

$$\limsup_{i\to\infty} \left\| p_i^{(k)} a - a p_i^{(k)} \right\| \leq 1/2^k, \quad a \in \mathcal{F}.$$

Let $(\mathcal{G}_n)$ be an increasing family of finite subsets of $\bigoplus_4 \mathcal{K}$ with dense union. Since $(p_i^{(1)})$ is an approximate unit of $\bigoplus_4 \mathcal{K}$, there is a finite rank projection $p_{i_k}^{(k)}$ such that

$$\left\| b - p_{i_k}^{(k)} b \right\| < 1/2^k \quad \text{for any } b \in \mathcal{G}_k,$$

and

$$\left\| p_{i_k}^{(k)} a - a p_{i_k}^{(k)} \right\| \leq 1/2^k \quad \text{for any } a \in \mathcal{F}.$$

Then, denote by $p_k = p_{i_k}^{(k)}$, and the sequence of finite rank projections $(p_k;\ k \in \mathbb{N})$ is an approximate unit of $\bigoplus_4 \mathcal{K}$ with

$$\lim_{k\to\infty} \|p_k a - a p_k\| \leq \lim_{k\to\infty} 1/2^k = 0.$$

This proves the statement for essential extensions.

If the extension is not essential, set $I = \bigoplus_4 \mathcal{K}$, and denote by $I^\perp$ the closed two-sided ideal of annihilators of I. Then, we have $I \cap I^\perp = \{0\}$. Consider the extension $\mathcal{C}/I^\perp$ of I by $D/I^\perp$. Note that this extension is essential. Since $(D \otimes \mathrm{M}_m(\mathbb{C}))/I^\perp$ is a proper quotient of $D \otimes \mathrm{M}_m(\mathbb{C})$, the abelian group $\mathrm{K}_0((D \otimes \mathrm{M}_m(\mathbb{C}))/I^\perp)$ is free and $\mathrm{K}_1((D \otimes \mathrm{M}_m(\mathbb{C}))/I^\perp) = 0$. It follows from the UCT that the extension $\mathcal{C}/I^\perp$ is trivial.

Applying the conclusion for essential extensions, we have that there is an approximate unit (p_i) of I such that

$$\lim_{i\to\infty} \|p_i \pi(a) - \pi(a) p_i\| = 0 \quad \text{for any } a \in \mathcal{F},$$

where π is the quotient map from $\mathcal{C}$ to $\mathcal{C}/I^\perp$. Since the restriction of π to I is an isometry, we have

$$\|p_i a - a p_i\| = \|\pi(p_i a - a p_i)\| = \|p_i \pi(a) - \pi(a) p_i\|,$$

and hence for any $a \in \mathcal{F}$,

$$\lim_{i\to\infty} \|p_i a - a p_i\| = \lim_{i\to\infty} \|p_i \pi(a) - \pi(a) p_i\| = 0,$$

as desired. □

Lemma 8.32. *The dimension drop circle C*-algebra described in (8.10) is the universal C*-algebra generated by z and w with the following relations:*

(1) $zz^* = z^*z = 1$;
(2) $zw = wz$, and $zw^* = w^*z$;
(3) $ww^* \perp w^*w$; and
(4) $ww^* + w^*w = 2 - z - z^*$.

Proof. Denote by $\mathcal{W}$ the universal C*-algebra generated by the generators and relations above. Consider the elements

$$z' = \left\{\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right\} \quad \text{and} \quad w' = \left\{\lambda \mapsto \begin{pmatrix} 0 & \sqrt{2(1-\Re\lambda)} \\ 0 & 0 \end{pmatrix}\right\}$$

in the concrete dimension drop circle algebra D. Then, z' and w' satisfy the relations of the lemma, and they generate D. Hence there is a surjective map from $\mathcal{W}$ to D.

Let us calculate the irreducible representations of $\mathcal{W}$.

Let π be an irreducible representation of $\mathcal{W}$, and let us still use the same notation for the images of the generators. Since z is a central unitary, we have that $z = \lambda\mathbf{1}$ for some $\lambda \in \mathbb{T}$.

If $\lambda = 1$, then,

$$ww^* + w^*w = 2 - z - z^* = 0,$$

and hence $w = 0$. Therefore, the representation π is one dimensional, and $\pi(z) = 1$ and $\pi(w) = 0$.

If $\lambda \neq 1$, the consider the element $v := w/\sqrt{2(1-\Re\lambda)}$. We then have

$$vv^* + v^*v = \mathbf{1} \quad \text{and} \quad vv^* \perp v^*v.$$

Therefore, the element v is an partial isometry generating a 2×2 matrix algebra. Thus, the representation is two dimensional, and

$$\pi(w) = \sqrt{2(1-\Re\lambda)}v \quad \text{and} \quad \pi(z) = \lambda(vv^* + v^*v).$$

Therefore, the irreducible representations of D exhaust the irreducible representations of $\mathcal{W}$, and hence $\mathcal{W} = D$, as desired. □

Remark 8.33. Since dimension drop circle algebras are semiprojective (Theorem 6.22 of [ELP98]), the relations of Lemma 8.32 are stable.

Lemma 8.34. *Consider the dimension drop circle C*-algebra D described in (8.10). Assume that the index map associated with an extension*

$$0 \longrightarrow \bigoplus_4 \mathcal{K} \longrightarrow \mathcal{C} \xrightarrow{\pi} D \otimes \mathrm{M}_m(\mathbb{C}) \longrightarrow 0$$

is zero. Then, there is a homomorphism $\eta : D \otimes \mathrm{M}_m(\mathbb{C}) \to \mathcal{C}$ such that $\pi \circ \eta = \mathrm{id}$.

Proof. Since the index map is zero and D has stable rank one, the C*-algebra $\mathcal{C}$ has stable rank one (see, for example, Proposition 4 of [LR95]).

Denote by (f_{ij}) a system of matrix units of $\mathrm{M}_m(\mathbb{C})$. Then there is a system of matrix units (e_{ij}) in $\mathcal{C}$ which lifts (f_{ij}). Consider the C*-algebra $e_{11}\mathcal{C}e_{11}$. It is an extension of $e_{11}(\bigoplus_4 \mathcal{K})e_{11}$—which is isomorphic to $\bigoplus_4 \mathcal{K}$—by D. Note that since $e_{11}\mathcal{C}e_{11}$ has stable rank one, any unitary in D lifts to a unitary in $e_{11}\mathcal{C}e_{11}$ (again, see, for example, Proposition 4 of [LR95]).

Denote by $\mathcal{C}' = e_{11}\mathcal{C}e_{11}$, and denote by z and w the generators of D specified by Lemma 8.32. Let us show that z and w can be lifted approximately, as then, by the semiprojectivity of D, there is a lifting of D to $\mathcal{C}'$.

Pick a unitary $Z \in \mathcal{C}'$ which lifts z, and pick any $W \in \mathcal{C}'$ which lifts w. By Lemma 8.31, there is an approximate unit (p_i) of $\bigoplus_4 \mathcal{K}$ such that each p_i is a projection and

$$\lim_{i\to\infty} \|p_i Z - Z p_i\| = \lim_{i\to\infty} \|p_i W - W p_i\| = 0.$$

Consider the elements

$$Z_i := (1-p_i)Z(1-p_i) + p_i$$

and

$$W_i := W - p_i W p_i,$$

and let us verify that the relations of Lemma 8.32 are satisfied approximately if p_i is sufficiently large.

For Condition (1),

$$\lim_{i\to\infty} Z_i Z_i^* = \lim_{i\to\infty} (1-p_i)Z(1-p_i)Z^*(1-p_i) + p_i = \mathbf{1}$$

and

$$\lim_{i\to\infty} Z_i^* Z_i = \lim_{i\to\infty} (1-p_i)Z^*(1-p_i)Z(1-p_i) + p_i = \mathbf{1}.$$

For Condition (2),

$$\begin{aligned} \lim_{i\to\infty} Z_i W_i - W_i Z_i &= \lim_{i\to\infty} ((ZW - WZ) - p_i(ZW - WZ)p_i) \\ &= 0 \quad (\text{since } (ZW - WZ) \in \bigoplus_4 \mathcal{K}), \end{aligned}$$

and

$$\begin{aligned} \lim_{i\to\infty} Z_i W_i^* - W_i^* Z_i &= \lim_{i\to\infty} ((ZW^* - W^*Z) - p_i(ZW^* - W^*Z)p_i) \\ &= 0 \quad (\text{since } (ZW^* - W^*Z) \in \bigoplus_4 \mathcal{K}). \end{aligned}$$

For Condition (3),

$$\begin{aligned} \lim_{i\to\infty} (W_i W_i^*)(W_i^* W_i) &= \lim_{i\to\infty} (WW^*W^*W - p_i(WW^*W^*W)p_i) \\ &= 0 \quad (\text{since } (WW^*W^*W) \in \bigoplus_4 \mathcal{K}). \end{aligned}$$

And for Condition (4),

$$\begin{aligned} & \lim_{i\to\infty}(W_iW_i^* + W_i^*W_i - (2 - Z_i - Z_i^*)) \\ = \ & \lim_{i\to\infty}((WW^* + W^*W - (2 - Z - Z^*)) - p_i(WW^* + W^*W - (2 - Z - Z^*))p_i) \\ = \ & 0 \quad (\text{since } (WW^* + W^*W - (2 - Z - Z^*)) \in \bigoplus_4 \mathcal{K}). \end{aligned}$$

Therefore, the C*-algebra D can be lifted to $\mathcal{C}'$, and hence there is a lifting $\eta : D\otimes \mathrm{M}_m(\mathbb{C}) \to \mathcal{C}$, as desired. ☐

Lemma 8.35. *Consider the dimension drop circle C*-algebra D described in (8.10). If the index map of an extension*

$$0 \longrightarrow \bigoplus\nolimits_4 \mathcal{K} \longrightarrow \mathcal{C} \xrightarrow{\pi} D \otimes \mathrm{M}_m(\mathbb{C}) \longrightarrow 0$$

is zero, then for any finite subset $\mathcal{F} \subset \mathcal{C}$ and any $\varepsilon > 0$, there is a unital sub-C-algebra $F \subseteq \mathcal{C}$ such that*

$$F \cong (D \otimes \mathrm{M}_m(\mathbb{C})) \oplus (\bigoplus_{i=1}^{4} \mathrm{M}_{k_i}(\mathbb{C}))$$

and $\mathcal{F} \subseteq_\varepsilon F$.

Proof. Since $D\otimes \mathrm{M}_m(\mathbb{C})$ is semiprojective, there is a finite set $\mathcal{G}$ of generators of $D\otimes \mathrm{M}_m(\mathbb{C})$ such that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if a linear map $\psi' : D \otimes \mathrm{M}_m(\mathbb{C}) \to \mathcal{C}$ is $\mathcal{G}$–$\delta(\varepsilon)$ multiplicative and $\pi \circ \psi'(g) = g$ for any $g \in \mathcal{G}$, then there is a homomophism $\psi : D\otimes \mathrm{M}_m(\mathbb{C}) \to \mathcal{C}$ such that

$$\|\psi(g) - \psi'(g)\| < \varepsilon$$

for any $g \in \mathcal{G}$ and $\pi \circ \psi = \mathrm{id}$. We may assume that $\mathcal{G}$ is in the unit ball of $D \otimes \mathrm{M}_m(\mathbb{C})$. Fix this set of generators.

By Lemma 8.34, there is an approximate lifting $\eta : D\otimes \mathrm{M}_m(\mathbb{C}) \to \mathcal{C}$. Without loss of generality, we may assume that $\mathcal{F} = \mathcal{F}_1 + \eta(\mathcal{F}_2)$, where $\mathcal{F}_1$ is a finite subset of $\bigoplus_4 \mathcal{K}$ and $\mathcal{F}_2$ is the set of the generators for $D \otimes \mathrm{M}_m(\mathbb{C})$.

By Lemma 8.31, there is an approximate unit (p_i) of $\bigoplus_4 \mathcal{K}$ such that

$$\lim_{i\to\infty} \|p_i d - dp_i\| = 0, \quad \forall d \in \eta(\mathcal{F}_2).$$

Then, there exists a natural number N such that for any $a \in \mathcal{F}$ and for any $i > N$,

$$\|a - (p_ia_1p_i + p_ia_2p_i + (1 - p_i)a_2(1 - p_i))\| < \varepsilon/2$$

for some $a_1 \in \mathcal{F}_1$ and $a_2 \in \eta(\mathcal{F}_2)$, and the map

$$D \otimes \mathrm{M}_m(\mathbb{C}) \ni f \mapsto (1 - p_i)\eta(f)(1 - p_i) \in (1 - p_i)\mathcal{C}(1 - p_i)$$

is $\mathcal{F}_2$–*delta*$(\varepsilon/2)$ multiplicative. Therefore, there is a homomorphism

$$\phi_i : D \otimes \mathrm{M}_m(\mathbb{C}) \to (1 - p_i)\mathcal{C}(1 - p_i)$$

such that for any $d \in \mathcal{F}_2$,

$$\|\phi_i(d) - (1 - p_i)\eta(f)(1 - p_i)\| < \varepsilon/2,$$

and

$$\pi \circ \phi_i = \mathrm{id}.$$

Thus, for any $a \in \mathcal{F}$,

$$a \in_\varepsilon p_i \mathcal{C} p_i \oplus \phi_i(D \otimes \mathrm{M}_m(\mathbb{C})).$$

Since $\pi \circ \phi_i = \mathrm{id}$, the map ϕ_i is injective. Note that $p_i \mathcal{C} p_i \cong \bigoplus_{i=1}^4 \mathrm{M}_{k_i}(\mathbb{C})$ for some k_i. Then

$$F := p_i \mathcal{C} p_i \oplus \phi_i(D \otimes \mathrm{M}_m(\mathbb{C}))$$

is the desired sub-C*-algebra. ☐

Remark 8.36. If the ideal of the extension is essential, then we can obtain the approximation structure more directly, and even in more general settings. For instance, we have the following lemma, which might be well known to experts.

Lemma 8.37. *Let D be a weakly semiprojective quasi-diagonal C*-algebra with the abelian group $K_0(D)$ free. Suppose that D satisfies the UCT. If the index map of an extension*

$$0 \longrightarrow \bigoplus_4 \mathcal{K} \longrightarrow \mathcal{C} \longrightarrow D \longrightarrow 0$$

is zero and the extension is essential, then for any finite subset $\mathcal{F} \subset \mathcal{C}$ and any ε, there is a unital sub-C-algebra*

$$F \cong D' \oplus (\bigoplus_{i=1}^4 \mathrm{M}_{k_i}(\mathbb{C}))$$

such that $\mathcal{F} \subseteq_\varepsilon F$, where D' is a quotient of D.

Proof. Since D is weakly semiprojective, there is a finite set $\mathcal{G}$ of generators of D such that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if a linear map $\psi' : D \to A$ is $\mathcal{G}$–$\delta(\varepsilon)$ multiplicative for some C*-algebra A, then there is a homomophism $\psi : D \to A$ such that

$$\|\psi(g) - \psi'(g)\| < \varepsilon$$

for any $g \in \mathcal{G}$. We may assume that $\mathcal{G}$ is in the unit ball of D. Fix this set of generators.

Since the index map is zero, and $\mathrm{K}_0(D)$ is free, the extension is trivial by the UCT. Denote by η the map from D to $\bigoplus_4 B(\mathscr{H}) \cong \mathcal{M}(\bigoplus_4 \mathcal{K})$. Without loss of generality, we may assume that $\mathcal{F} = \mathcal{F}_1 + \eta(\mathcal{F}_2)$, where $\mathcal{F}_1$ is a finite subset of $\bigoplus_4 \mathcal{K}$ and $\mathcal{F}_2$ is the set of generators for D.

We assert that for any $\varepsilon' > 0$, there is an approximate unit $\{p_i\}$ of $\bigoplus_4 \mathcal{K}$ with each p_i finite rank projection, and for sufficiently large i,

$$\|[p_i, a]\| < \varepsilon', \quad a \in \eta(\mathcal{F}_2).$$

Indeed, since D is quasi-diagonal, there is a faithful representation $\sigma : D \to \bigoplus_4 B(\mathscr{H})$ and finite rank projections $\{p_i\}$ converging to the identity strongly such that

$$\lim_{i\to\infty} [p_i, \pi(d)] = 0, \quad d \in D.$$

By taking infinite multiplicity, we may assume that $\pi(D)$ does not contain any compact operators. By Voiculescu's Theorem, the representation σ is approximately unitarily equivalent to η. Thus the assertion holds.

Applying the assertion to $\varepsilon' = \varepsilon/4$, we have an approximate unit (p_i) of $\bigoplus_4 \mathcal{K}$ with each p_i a finite rank projection, and for sufficiently large i,

$$\|[p_i, a]\| < \varepsilon/4, \quad a \in \eta(\mathcal{F}_2).$$

Therefore, there exists N such that for any $a \in \mathcal{F}$ and for any $i > N$,

$$\|a - (p_i a_1 p_i + p_i a_2 p_i + (1 - p_i)a_2(1 - p_i))\| < \varepsilon/2$$

for some $a_1 \in \mathcal{F}_1$ and $a_2 \in \eta(\mathcal{F}_2)$, and the map

$$D \ni f \mapsto (1 - p_i)\eta(f)(1 - p_i) \in (1 - p_i)\mathcal{C}(1 - p_i)$$

is $\mathcal{F}_2$–$\delta(\varepsilon/2)$ multiplicative. Therefore, there is a homomorphism $\phi_i : D \to (1-p_i)\mathcal{C}(1-p_i)$ such that for any $d \in \mathcal{F}_2$,

$$\|\phi_i(d) - (1 - p_i)\eta(f)(1 - p_i)\| < \varepsilon/2,$$

and hence for any $a \in \mathcal{F}$,

$$a \in_\varepsilon p_i\mathcal{C}p_i \oplus \phi_i(D).$$

Note that $p_i\mathcal{C}p_i \cong \bigoplus_{i=1}^4 \mathrm{M}_{k_i}(\mathbb{C})$ for some k_i. Then $F := p_i\mathcal{C}p_i \oplus \phi_i(D)$ is the desired C*-subalgebra. □

Corollary 8.38. *For any C*-algebra $\mathcal{B}_{n/m}$, there is a sub-C*-algebra $H_{n/m}$ such that $H_{n/m}$ is a direct sum of a dimension drop circle algebra and a finite dimensional C*-algebra, and*

$$\mathrm{dist}(a, H_{n/m}) < \frac{2\pi}{m},$$

where $a \in \{u, p, v, q\}$, the standard generators of $\mathcal{B}_{n/m}$.

Proof. By Lemma 8.35, the C*-algebra $\mathcal{C}_{n/m}$ can be locally approximated by direct sums of dimension drop circle algebras and finite dimensional C*-algebras. The corollary follows by Lemma 8.29. □

Theorem 8.39. *The extended rotation algebra $\mathcal{B}_\theta$ is AF for generic θ.*

Proof. Let us first show that $\mathcal{B}_\theta$ is an inductive limit of dimension drop circle algebras, for θ belonging to a dense G_δ set. Consider the sections $\mathcal{G} = \{(u_\theta), (p_\theta), (v_\theta), (q_\theta)\}$. Note that each fibre C*-algebra is generated by the restriction of $\mathcal{G}$. Therefore, in order to prove the theorem, it is enough to show that the restriction of $\mathcal{G}$ to the generic fibre can be approximated by dimension drop circle C*-algebras.

For each rational fibre $\mathcal{B}_{n/m}$, by Corollary 8.38, there is a dimension drop circle C*-algebra $H_{n/m} \subseteq \mathcal{B}_{n/m}$ such that the distance of each element of the restriction of $\mathcal{G}$ from $H_{n/m}$ is strictly less than $2\pi/m$. Since $H_{n/m}$ is semiprojective and the field $(\mathcal{B}_\theta)$ is upper semicontinuous, there is an open neighborhood $U_{n/m}$ of n/m such that for any $\theta \in U_{n/m}$, there is a dimension drop circle C*-algebra $H_{n/m,\theta} \subseteq \mathcal{B}_\theta$ such that the distance of each element of the restriction of $\mathcal{G}$ to θ from $H_{n/m,\theta}$ is still strictly less than $2\pi/m$.

For each natural number k, set

$$U_k = \bigcup_{\substack{n/m \\ m>k}} U_{n/m}.$$

Since the rational numbers with denominator greater than k are dense, the open set U_k is dense. By the Baire Category Theorem, the G_δ set

$$U = \bigcap_k U_k$$

is a dense. As we shall now show, $\mathcal{B}_\theta$ is a limit of dimension drop circle algebras for any $\theta \in U$.

For any $\theta \in U$, let us show that $\mathcal{B}_\theta$ can be locally approximated by dimension drop circle algebras. For any $\varepsilon > 0$, we have that $\theta \in U_k$ for any natural number k, and in particular, $\theta \in U_k$ for $k > [2\pi/\varepsilon]$, where $[2\pi/\varepsilon]$ denotes the integer part of $2\pi/\varepsilon$. Therefore, we have that $\theta \in U_{n/m}$ for some rational number n/m with $m > k > [2\pi/\varepsilon]$. By the construction of $U_{n/m}$, there is a dimension drop circle C*-algebra $H_{n/m,\theta} \subseteq \mathcal{B}_\theta$ with the distance between $H_{n/m,\theta}$ and the restrictions of $\mathcal{G}$ to θ less that $2\pi/m$, and hence less that ε.

By Theorem 6.22 of [ELP98], the C*-algebra $\mathcal{B}_\theta$ belongs to the class of simple unital inductive limit of dimension drop circle C*-algebras, a class which contains the simple AF-algebras and the objects of which are classified by their order-unit K_0-group paired with the simplex of tracial states, together with the K_1-group (see [Tho97]). By Remark 8.28, the C*-algebra $\mathcal{B}_\theta$ has trivial K_1-group, and has the ordered K_0-group and paired trace simplex of an AF-algebra; hence it is isomorphic to that AF-algebra, as desired. □

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