NUCLEARITY AND WEAK UNIQUENESS

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Abstract. We characterize nuclearity using a uniqueness theorem. More precisely: Let $\mathcal{A}$ be a unital separable simple C^*-algebra. Then the following are equivalent:

(1) $\mathcal{A}$ is nuclear
(2) For every unital separable C^*-algebra $\mathcal{C}$, for all unital injective $*$-homomorphisms $\phi, \psi : \mathcal{C} \to \mathcal{A}$ such that $Cu(\phi) = Cu(\psi)$, ϕ and ψ are weakly approximately unitarily equivalent.

Here, for a C^*-algebra $\mathcal{D}$, $Cu(\mathcal{D})$ is the Cuntz semigroup of $\mathcal{D}$.

1. Introduction

In C^*-algebra theory, a *uniqueness theorem* roughly says the following: If $\phi, \psi : \mathcal{A} \to \mathcal{C}$ are two maps between C^*-algebras ($\mathcal{A}$ and $\mathcal{C}$) which have the same K-theory, then ϕ and ψ are approximately unitarily equivalent.

In C^*-algebra theory, uniqueness theorems play interesting roles in both extension theory (see, for example, [1], [5], [16], [25], [36], [50]) and the Elliott classification program for simple nuclear C^*-algebras using K-theory invariants (see, for example, [15], [16], [18], [20], [28], [29], [36])

The class of nuclear C^*-algebras has been a centre of interest for C^*-algebraists since the very important work of Choi and Effros, Connes, and Effros and Lance (see, for example, [8], [6], [7], [10], [19]). The class of nuclear C^*-algebras is a very large class, including many interesting C^*-algebras coming from group representation theory, dynamical systems and mathematical physics (see, for example, [2], [10], [12], [14], [16], [18], [22], [27], [42]). This class has many nice K-theoretic properties (see, for example, [2], [7]), and is the main object of study in the Elliott classification program.

In this paper, we characterize nuclearity by using a uniqueness theorem. In the uniqueness theorem for our characterization, the invariant that we use is the *Cuntz semigroup*, which has recently been of much interest in the classification program (see, for example, [4], [9], [11], [44], [49]). Among other things, it has been used to distinguish between nonisomorphic simple unital AH-algebras whose usual K-theory invariants are the same (see [49]). See below (in this section) for more information about the Cuntz semigroup.

We note that in standard uniqueness theorems, approximate unitary equivalence in norm is used. However, to achieve our characterization of nuclearity, we require a weaker type of approximate unitary equivalence.

Definition 1.1. *Let $\mathcal{A}, \mathcal{C}$ be C^*-algebras, with $\mathcal{C}$ unital. Let $\phi, \psi : \mathcal{A} \to \mathcal{C}$ be two $*$-homomorphisms.*

(1) *ϕ and ψ are said to be* approximately unitarily equivalent *if there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of unitaries in $\mathcal{C}$ such that for all $a \in \mathcal{A}$,*

$$u_n \psi(a)(u_n)^* \to \phi(a)$$

in the norm topology on $\mathcal{C}$.

(2) *ϕ and ψ are said to be* weakly approximately unitarily equivalent *if there exist two nets $\{u_\alpha\}_{\alpha \in I}$ and $\{w_\beta\}_{\beta \in J}$ of unitaries in $\mathcal{C}$ such that for all $a \in \mathcal{A}$,*

$$u_\alpha \psi(a)(u_\alpha)^* \to \phi(a)$$

and

$$w_\beta \phi(a)(w_\beta)^* \to \psi(a)$$

in the relative weak topology on $\mathcal{C}$.
(The relative weak topology on $\mathcal{C}$ is the weak topology on $\mathcal{C}$ induced by the continuous linear functionals in $\mathcal{C}^*$.)

(3) *If, in addition, $\mathcal{C}$ is a von Neumann algebra, then we say that ϕ and ψ are* weak* approximately unitarily equivalent *if the limits in (2) hold in the weak* topology on $\mathcal{C}$.*
(In other words, we restrict to the normal linear functionals in the predual $\mathcal{C}_*$.)

These weaker notions of approximate unitary equivalence have been previously defined and studied. (See [1], [16] Theorem II.5.8, [17] and [31]. We note that except for [17], all the papers here mentioned concern maps with codomain $\mathbb{B}(\mathcal{H})$. The relationship between our work and [17] is Lemma 3.6.)

Our characterization of nuclearity is the following:

Theorem 1.1. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra. Then the following statements are equivalent:*

(1) *$\mathcal{A}$ is nuclear.*
(2) *For every unital separable C^*-algebra $\mathcal{C}$, for all unital injective $*$-homomorphisms $\phi, \psi : \mathcal{A} \to \mathcal{C}$, if $Cu(\phi) = Cu(\psi)$ then ϕ and ψ are weakly approximately unitarily equivalent.*

The arguments in this paper combine von Neumann algebra theory and C^*-algebra theory (especially classification theory - a good reference is [36]). We extensively use ideas from the theory of absorbing extensions - especially the ideas of [25]. We also use ideas from the paper of [17]. Finally, good references for von Neumann algebra theory are [33] and [40], [47].

We end this introduction by giving some basic results unfamiliar to nonexperts, and fixing some notation that will be used throughout this paper.

Let $\mathcal{C}$ be a C^*-algebra and let $\rho \in \mathcal{C}^*$ be a bounded linear functional. We let $\|.\|_\rho$ be the seminorm on $\mathcal{C}$ that is given by $\|c\|_\rho =_{df} |\rho(c)|$ for all $c \in \mathcal{C}$. If ρ is positive, we let $\|.\|_{2,\rho}$ and $\|.\|_{2,\rho}^\sharp$ be the seminorms on $\mathcal{C}$ that are given by $\|c\|_{2,\rho} =_{df} \rho(c^*c)^{1/2}$ and $\|c\|_{2,\rho}^\sharp =_{df} \|c\|_{2,\rho} + \|c^*\|_{2,\rho}$ for all $c \in \mathcal{C}$.

Next, we need some results about the Cuntz semigroup and comparison of positive elements which (until recently) might not have been well-known to nonexperts. For the convenience of the reader, we provide the (short) exposition of Rordam in the beginning of section 4 of [44] (with slight modifications). Let $\mathcal{C}$ be a C^*-algebra. In [13], Cuntz associates to $\mathcal{C}$ a positive ordered abelian semigroup $Cu(\mathcal{C})$

in the following manner: Let $\mathbb{M}_\infty(\mathcal{C})_+$ denote the (disjoint) union $\bigcup_{n=1}^\infty \mathbb{M}_n(\mathcal{C})_+$. For $a \in \mathbb{M}_n(\mathcal{C})_+$ and $b \in \mathbb{M}_m(\mathcal{C})_+$, set $a \oplus b = diag(a, b) \in \mathbb{M}_{n+m}(\mathcal{C})_+$, and write "$a \preceq b$" if there is a sequence $\{x_k\}$ in $\mathbb{M}_{m,n}(\mathcal{C})$ such that $(x_k)^* b x_k \to a$. (Note that in these computations, we can replace both m and n by $max\{m, n\}$. i.e., we can restrict to square matrices.) We say that a is *Cuntz equivalent* to b (and write "$a \sim b$") if $a \preceq b$ and $b \preceq a$. For $a \in \mathbb{M}_\infty(\mathcal{C})_+$, let $[a]$ denote the equivalence class of a under $\sim$; i.e., $[a] = \{b \in \mathbb{M}_\infty(\mathcal{C})_+ : b \sim a\}$. The *Cuntz semigroup* of $\mathcal{C}$ is defined to be $Cu(\mathcal{C}) =_{df} \mathbb{M}_\infty(\mathcal{C})/\sim$ given the relations:

i. $[a] + [b] = [a \oplus b]$
ii. $[a] \leq [b]$ if and only if $a \preceq b$

for all $a, b \in \mathbb{M}_\infty(\mathcal{C})_+$. The Cuntz semigroup $Cu(\mathcal{C})$ is a positive ordered abelian semigroup.

For a positive element a of a C^*-algebra $\mathcal{C}$, for $\epsilon > 0$, we let $(a - \epsilon)_+$ be the positive element of $\mathcal{C}$ that is given by $f_\epsilon(a)$, where f_ϵ is the continuous function that is given by $f_\epsilon(t) =_{df} \max\{t - \epsilon, 0\}$. Also, for positive elements $a, b \in \mathcal{C}$, we write $a \approx b$ when there exists $x \in \mathcal{C}$ such that $a = x^*x$ and $b = xx^*$. We will also need the following facts about comparison of positive elements in a C^*-algebra, which can be found in [13] and [43] section 2:

For a C^*-algebra $\mathcal{C}$, for positive elements $a, b \in \mathcal{C}$, for strictly positive real numbers $\epsilon_1, \epsilon_2 > 0$, we have the following:

(1) $a \preceq b$ if and only if $(a - \epsilon)_+ \preceq b$ for all $\epsilon > 0$
(2) $a \preceq b$ if and only if for every $\epsilon > 0$, there exists $x \in \mathcal{C}$ such that $x^* b x = (a - \epsilon)_+$.
(3) $a \preceq b$ if and only if for every $\epsilon > 0$ there exists a positive element $c \in Her(b)$ such that $(a - \epsilon)_+ \approx c$.
(4) Let p, q be projections in $\mathcal{C}$ then $p \preceq q$ if and only if p is Murray-von Neumann equivalent to a subprojection of q.
(5) For every $\epsilon > 0$, if $\|a - b\| < \epsilon$ then $(a - \epsilon)_+ \preceq b$.

(Note that for an arbitrary C^*-algebra $\mathcal{C}$, for projections p, q in $\mathcal{C}$, $p \sim q$ (i.e., p being Cuntz equivalent to q) is not necessarily the same as p and q being Murray-von Neumann equivalent. For example, if $\mathcal{C}$ is simple purely infinite then $p \sim q$ for all nonzero projections $p, q \in \mathcal{C}$. However, when $\mathcal{C}$ is a von Neumann algebra, Cuntz equivalence and Murray-von Neumman equivalence for projections are indeed the same, by [33] Proposition 6.2.4.)

Next, we fix some notation for tensor products. For C^*-algebras $\mathcal{A}, \mathcal{B}$ (which could be von Neumann algebras!), $\mathcal{A} \otimes \mathcal{B}$ will always be the minimal C^*-algebraic tensor product of $\mathcal{A}$ and $\mathcal{B}$. For C^*-algebras $\mathcal{A}, \mathcal{B}$, $\mathcal{A} \odot \mathcal{B}$ is the algebraic tensor product of $\mathcal{A}$ and $\mathcal{B}$. For von Neumann algebras $\mathcal{M}, \mathcal{N}$, $\mathcal{M} \overline{\otimes} \mathcal{N}$ will always be the von Neumann algebraic tensor product of $\mathcal{M}$ and $\mathcal{N}$.

Finally, for C^*-algebras $\mathcal{C}$ and $\mathcal{D}$, a $*$-homomorphism $\phi : \mathcal{C} \to \mathcal{D}$ naturally induces a $*$-homomorphism $\phi \otimes 1_{\mathbb{M}_n} : \mathbb{M}_n(\mathcal{C}) = \mathcal{C} \otimes \mathbb{M}_n \to \mathbb{M}_n(\mathcal{D}) = \mathcal{D} \otimes \mathbb{M}_n$ for every $n \geq 1$. Moreover, for every $n \geq 1$, for $a, b \in \mathbb{M}_n(\mathcal{C})_+$, $a \sim b$ implies that $(\phi \otimes 1_{\mathbb{M}_n})(a) \sim (\phi \otimes 1_{\mathbb{M}_n})(b)$, and we get a natural ordered semigroup homomorphism $Cu(\phi) : Cu(\mathcal{C}) \to Cu(\mathcal{D})$.

2. Properly infinite von Neumann algebra codomains

Definition 2.1. *Let $\mathcal{A}$, $\mathcal{C}$ be two C^*-algebras. A subset $S \subseteq \mathcal{C}$ is said to be* full *if every element $c \in S$ is a norm-full element of $\mathcal{C}$ (i.e., c is not contained in any proper two-sided C^*-ideal of $\mathcal{C}$). A map $\phi : \mathcal{A} \to \mathcal{C}$ is said to be* full *if $\phi(\mathcal{A}) - \{0\}$ is a full subset of $\mathcal{C}$. (Note that if ϕ is linear, then fullness of ϕ implies injectivity of ϕ.)*

We will need the following excision of pure states result whose proof can be found in, among other places, [25] Lemma 8 or [36] Lemma 5.3.2:

Lemma 2.1. *Let $\mathcal{A}$ be a unital separable C^*-algebra. Let ρ be a pure state on $\mathcal{A}$. Then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of positive elements in $\mathcal{A}$ with $\|a_n\| = 1$ for all n, such that*

$$lim_{n\to\infty} \|a_n(a - \rho(a))a_n\| = 0.$$

for all $a \in \mathcal{A}$.

The next lemma follows from the arguments of [25] Lemma 10 combined with [33] Lemma 6.3.3. (See also [30] Proposition 2.1.)

Lemma 2.2. *Let $\mathcal{M}$ be a properly infinite von Neumann algebra and let $\mathcal{A}$ be a unital separable C^*-algebra. Let $n \geq 1$ be a positive integer, and let $\sigma : \mathcal{A} \to \mathbb{M}_n(\mathbb{C})$ and $\eta : \mathbb{M}_n(\mathbb{C}) \to \mathcal{M}$ be unital completely positive maps. Let $\psi : \mathcal{A} \to \mathcal{M}$ be the unital completely positive map given by*

$$\psi =_{df} \eta \circ \sigma$$

Then ψ can be approximated in the pointwise-norm operator topology by finite sums of maps of the form

$$a \mapsto m\rho_n((a_0)^* a a_0)m^*$$

where ρ is a pure state on $\mathcal{A}$, $\rho_n =_{df} \rho \otimes id_{\mathbb{M}_n(\mathbb{C})} : \mathbb{M}_n(\mathcal{A}) \to \mathbb{M}_n(\mathbb{C})$ is the natural map induced by ρ, m is a row matrix in $\mathcal{M}^n$ and a_0 is a row matrix in $\mathcal{A}^n$.

Lemma 2.3. *Let $\mathcal{M}$ be a properly infinite von Neumann algebra, and let $\mathcal{A}$ be a separable full unital C^*-subalgebra of $\mathcal{M}$. Let $\sigma : \mathcal{A} \to \mathbb{M}_n(\mathbb{C})$ and $\eta : \mathbb{M}_n(\mathbb{C}) \to \mathcal{M}$ be two unital completely positive maps. Let $\psi : \mathcal{A} \to \mathcal{M}$ be the unital completely positive map given by*

$$\psi =_{df} \eta \circ \sigma$$

Then ψ can be approximated in the pointwise-norm topology by maps of the form

$$a \mapsto xax^*.$$

where x is an element of $\mathcal{M}$.

Proof. Firstly, let $\epsilon > 0$ be given and let $\mathcal{F}$ be a finite subset of $\mathcal{A}$. We will approximate ψ on $\mathcal{F}$ in the norm topology. For simplicity, we may assume that the elements of $\mathcal{F}$ all have norm less than or equal to one. By Lemma 2.2, let $\rho^1, \rho^2, ..., \rho^k$ be a finite set of pure states on $\mathcal{A}$, let $m_1, m_2, ..., m_k$ be a set of row matrices in $\mathcal{M}^n$ and let $a_1, ..., a_k$ be a set of row matrices in $\mathcal{A}^n$ such that on $\mathcal{F}$, ψ is within $\epsilon/100$ of the map

$$a \mapsto \sum_{i=1}^{k} m_i {\rho^i}_n((a_i)^* a a_i)(m_i)^*.$$

By Lemma 2.1, for each i, let c_i be a positive element of $\mathcal{A}$ with $\|c_i\| = 1$ such that

$$diag(c_i, c_i, ..., c_i){\rho^i}_n((a_i)^* a a_i) diag(c_i, c_i, ..., c_i)$$

is within $\epsilon/(100k(2\|m_i\|^2 + 1))$ of

$$diag(c_i, c_i, ..., c_i)(a_i)^* a a_i diag(c_i, c_i, ..., c_i)$$

for every $a \in \mathcal{F}$. Here, $diag(c_i, c_i, ..., c_i)$ is the element of $\mathbb{M}_n(\mathcal{A})$ with c_is in the diagonal and zeroes everywhere else.

Since each c_i has norm one, since $\mathcal{A}$ is a full C^*-subalgebra of $\mathcal{M}$, since $\mathcal{M}$ is a properly infinite von Neumann algebra, and by [33] Lemma 6.3.3 (see also [36] Lemma 3.5.8), for each i, let x_i be an element of $\mathcal{M}$ with norm less than 5/4 such that for every $a \in \mathcal{F}$,

(a) $x_i (c_i)^2 (x_i)^* = 1_{\mathcal{M}}$ for each i; and
(b) $\|x_i c_i (a_i)^* a a_j c_j (x_j)^*\| < \epsilon/(100(k\|m_i\|\|m_j\| + 1))$ for $i \neq j$.

Take $x =_{df} \sum_{i=1}^k m_i x_i c_i (a_i)^*$. Then on $\mathcal{F}$, ψ is within ϵ of the map $a \mapsto xax^*$. □

We will need the following perturbation argument for isometries:

Lemma 2.4. *For every $\epsilon > 0$, there exists $\delta > 0$ such that for every unital C^*-algebra $\mathcal{C}$, if $x \in \mathcal{C}$ satisfies*

$$\|x^* x - 1\| < \delta$$

then there exists an isometry $v \in \mathcal{C}$ such that

$$\|v - x\| < \epsilon$$

Lemma 2.5. *Let $\mathcal{M}$ be a properly infinite von Neumann algebra. Let $\rho_1, \rho_2, ..., \rho_k$ be positive elements of the predual $\mathcal{M}_*$. Let $\mathcal{A}$ be a separable simple nuclear unital C^*-subalgebra of $\mathcal{M}$, and let $\phi : \mathcal{A} \to \mathcal{M}$ be a unital completely positive map. Then for every $\epsilon > 0$ and for every finite subset $\mathcal{F} \subseteq \mathcal{A}$, there exists a unitary $u \in \mathcal{M}$ such that*

$$\|\phi(a) - uau^*\|_{\rho_i} < \epsilon$$

for all $a \in \mathcal{F}$ and for $1 \leq i \leq k$.

Proof. Note that since $\mathcal{A}$ is a simple unital C^*-subalgebra of $\mathcal{M}$, $\mathcal{A}$ is full in $\mathcal{M}$.

Let $\epsilon > 0$ and a finite subset $\mathcal{F} \subseteq \mathcal{A}$ be given. Contracting $\epsilon > 0$ if necessary, we may assume that all the elements of $\mathcal{F}$ are positive elements with norm less than or equal to one, and we may also assume that each ρ_i has norm less than or equal to one. We may also assume that $1_{\mathcal{A}} \in \mathcal{F}$. We may also assume that $\epsilon < 1/100$. Apply Lemma 2.4 to get a strictly positive real number $\delta > 0$. Contracting δ if necessary, we may assume that $0 < \delta < \epsilon/100$. We denote the above statements by "(*)".

Since $\mathcal{A}$ is a nuclear C^*-algebra, let $n \geq 1$ be a positive integer, let $\sigma : \mathcal{A} \to \mathbb{M}_n(\mathbb{C})$ and $\eta : \mathbb{M}_n(\mathbb{C}) \to \mathcal{M}$ be unital completely positive maps such that the unital completely positive map $\psi : \mathcal{A} \to \mathcal{M}$ given by

$$\psi =_{df} \eta \circ \sigma$$

is within $\epsilon/100$ of ϕ on $\mathcal{F}$. In other words, for all $a \in \mathcal{F}$,

$$\|\phi(a) - \psi(a)\| < \epsilon/100$$

We denote the above inequality by "(**)".

By Lemma 2.3, let $x \in \mathcal{M}$ be such that for all $a \in \mathcal{F}$,

$$\|\psi(a) - x^*ax\| < \delta$$

We denote the above inequality by "(***)".

Since $1_{\mathcal{A}} \in \mathcal{F}$, and since ψ is a unital map, we have, by (***), that $\|1_{\mathcal{M}} - x^*x\| < \delta$. Hence, by the definition of δ and by Lemma 2.4, let $v \in \mathcal{M}$ be a partial isometry such that $\|v - x\| < \epsilon/100$. From this, (*), (**), and (***), it follows that for all $a \in \mathcal{F}$, we have that

$$\begin{aligned}
& \|\phi(a) - v^*av\| \\
\leq \; & \|\phi(a) - \psi(a)\| + \|\psi(a) - x^*ax\| + \|x^*ax - v^*av\| \\
< \; & \epsilon/100 + \epsilon/100 + \|x^*ax - v^*av\| \\
= \; & \epsilon/50 + \|x^*ax - v^*av\| \\
\leq \; & \epsilon/50 + \|x^*ax - v^*ax\| + \|v^*ax - v^*av\| \\
\leq \; & \epsilon/50 + \|x^* - v^*\|\|x\| + \|v^*\|\|x - v\| \\
= \; & \epsilon/50 + (\epsilon/100)(1 + \epsilon/100) + \epsilon/100 \\
= \; & \epsilon/50 + \epsilon/100 + \epsilon/10000 + \epsilon/100 \\
< \; & 2\epsilon/25
\end{aligned}$$

We denote the above computation by "(+)".

Since $\mathcal{M}$ is properly infinite, let $\{p_i\}_{i=1}^{\infty}$ be a sequence of pairwise orthogonal projections in $\mathcal{M}$ such that

(1) p_i is properly infinite for all $i \geq 1$
(2) p_i is Murray-von Neumann equivalent to $1_{\mathcal{M}}$ in $\mathcal{M}$ for all $i \geq 1$
(3) $1_{\mathcal{M}} = \sum_{i=1}^{\infty} p_i$, where the sum converges in the strong operator topology in $\mathcal{M}$.

We denote the above statements by "(++)".

Now for all $n \geq 1$, let $P_n =_{df} \sum_{i=1}^{n} p_i$. Therefore, $P_n \rightarrow 1_{\mathcal{M}}$ in the strong operator topology as $n \rightarrow \infty$. Hence, we must have that for all $b \in \mathcal{M}$, $P_n b P_n \rightarrow b$, and $(1 - P_n)b(1 - P_n) \rightarrow 0$ in the strong operator topology on $\mathcal{M}$ as $n \rightarrow \infty$. Hence, choose $N \geq 2$ such that for all $n \geq N$, for all $a \in \mathcal{F}$, for $1 \leq i \leq k$, the following statements hold:

(a) $\|v^*av - P_n v^*avP_n\|_{\rho_i} < min\{\epsilon/100, (\epsilon/100)^2\}$
(b) $\|1 - P_n\|_{\rho_i} < min\{\epsilon/100, (\epsilon/100)^2\}$

We denote the above statements by "(++)".

Note that by our choice of the p_ns, $1 - P_N$ (which contains p_{N+1}) is Murray-von Neumann equivalent to $1_{\mathcal{M}}$ in $\mathcal{M}$. Hence, since v is an isometry, $v(1 - P_N)v^*$ (i.e., the range projection of $v(1 - P_N)$ is Murray-von Neumann equivalent to $1_{\mathcal{M}}$ in $\mathcal{M}$. Hence, $(1_{\mathcal{M}} - vv^*) + v(1_{\mathcal{M}} - P_N)v^*$ is Murray-von Neumann equivalent to $1_{\mathcal{M}}$ in $\mathcal{M}$. So let w be a partial isometry in $\mathcal{M}$ with initial projection $1_{\mathcal{M}} - P_N$ and range projection $(1_{\mathcal{M}} - vv^*) + v(1_{\mathcal{M}} - P_N)v^*$. (In other words, $w^*w = 1_{\mathcal{M}} - P_N$ and $ww^* = (1_{\mathcal{M}} - vv^*) + v(1_{\mathcal{M}} - P_N)v^*$.) Let $u \in \mathcal{M}$ be the unitary given by

$$u =_{df} w + vP_N$$

Hence, since $\mathcal{F}$ consists of positive elements with norm less than or equal to one, and since for $1 \leq i \leq k$, ρ_i is a positive linear functional with norm less than or equal to one, and by (++), for $1 \leq i \leq k$, for $a \in \mathcal{F}$, we have that

$$\begin{aligned} & \|w^*aw\|_{\rho_i} \\ = \ & |\rho_i(w^*aw)| \\ \leq \ & |\rho_i(w^*w)| \\ = \ & |\rho_i(1_{\mathcal{M}} - P_N)| \\ = \ & \|1_{\mathcal{M}} - P_N\|_{\rho_i} \\ < \ & \epsilon/100 \end{aligned}$$

We denote the above computation by "(V)".

For the same reasons as (V), but also by the Cauchy-Schwarz inequality for positive linear functionals, for $1 \leq i \leq k$, for $a \in \mathcal{F}$, we have that

$$\begin{aligned} & \|w^*avP_N\|_{\rho_i} \\ = \ & |\rho_i(w^*avP_N)| \\ \leq \ & (\rho_i(w^*avP_Nv^*aw))^{1/2}(\rho_i(1))^{1/2} \\ \leq \ & |\rho_i(w^*w)^{1/2}| \\ = \ & |\rho_i(1_{\mathcal{M}} - P_N)^{1/2}| \\ = \ & \|1_{\mathcal{M}} - P_N\|_{\rho_i}^{1/2} \\ < \ & \epsilon/100 \end{aligned}$$

Note that in the above application of the Cauchy-Schwarz inequality, w^*avP_N is the variable on the left and $1_{\mathcal{M}}$ is the variable on the right. We denote the above computation by "(VV)".

For the same reasons as (VV), we have that for $1 \leq i \leq k$ and for $a \in \mathcal{F}$,

$$\begin{aligned} & \|P_Nv^*aw\|_{\rho_i} \\ = \ & |\rho_i(P_Nv^*aw)| \\ = \ & \rho_i(1)^{1/2}\rho_i(w^*avP_Nv^*aw)^{1/2} \\ \leq \ & \rho_i(w^*w)^{1/2} \\ = \ & \rho_i(1_{\mathcal{M}} - P_N)^{1/2} \\ = \ & \|1_{\mathcal{M}} - P_N\|_{\rho_i}^{1/2} \\ < \ & \epsilon/100 \end{aligned}$$

We denote the above computation by "(VVV)".

From (++), (V), (VV) and the definition of u, we have that for $1 \leq i \leq k$, for $a \in \mathcal{F}$,

$$\begin{aligned} & \|u^*au - v^*av\|_{\rho_i} \\ = \ & \|(w + vP_N)^*a(w + vP_N) - v^*av\|_{\rho_i} \\ = \ & \|w^*aw + w^*avP_N + P_Nv^*aw + P_Nv^*avP_N - v^*av\|_{\rho_i} \\ \leq \ & \|w^*aw\|_{\rho_i} + \|w^*avP_N\|_{\rho_i} + \|P_Nv^*aw\|_{\rho_i} + \|P_Nv^*avP_N - v^*av\|_{\rho_i} \\ < \ & \epsilon/100 + \epsilon/100 + \epsilon/100 + \epsilon/100 \\ = \ & \epsilon/25 \end{aligned}$$

Combining this computation with (+), since the ρ_i are states, and by the triangle inequality, we have that for $a \in \mathcal{F}$, for $1 \leq i \leq k$,

$$\|\phi(a) - u^*au\|_{\rho_i} < 2\epsilon/25 + \epsilon/25 < \epsilon$$

as required. □

Lemma 2.6. *Let $\mathcal{M}$ be a properly infinite von Neumann algebra. Let $\mathcal{A}$ be a unital simple separable nuclear C^*-algebra. Suppose that $\phi, \psi : \mathcal{A} \to \mathcal{M}$ are two unital $*$-homomorphisms (which will necessarily be injective). Then ϕ and ψ are weak* approximately unitarily equivalent.*

Proof. By symmetry, it suffices to find a net of unitaries in $\mathcal{M}$, such that conjugating ψ by these unitaries gives a net that converges to ϕ in the pointwise-weak* topology.

By identifying $\mathcal{A}$ with $\psi(\mathcal{A})$, we may assume that $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{M}$ and that ψ is the natural inclusion map.

But by Lemma 2.5, there exists a net $\{u_\alpha\}$ of unitaries in $\mathcal{M}$ such that for $a \in \mathcal{A}$, $\phi(a) = lim_\alpha (u_\alpha)^* a u_\alpha$ where the limit converges in the weak* topology, as required. □

3. Finite von Neumann algebra codomains

Recall the following (notation) from the introduction: For positive elements a, b of a C^*-algebra $\mathcal{C}$, we write "$a \approx b$" if there exists $x \in \mathcal{C}$ such that $x^*x = a$ and $xx^* = b$.

Lemma 3.1. *Let $\mathcal{M}$ be a finite von Neumann algebra. Let $a, b \in \mathcal{M}$ be positive elements with support projections p, q respectively. If $a \sim b$ (i.e., a and b are Cuntz equivalent in $\mathcal{M}$) then $p \sim q$ (i.e., p and q are Murray-von Neumann equivalent in $\mathcal{M}$).*

Proof. For simplicity, we may assume that both a and b have norm less than or equal to one.

By symmetry and since $\mathcal{M}$ is a von Neumann algebra (and by [33] Proposition 6.2.4), it suffices to show that $p \preceq q$ (i.e., that p is Murray-von Neumman equivalent to a subprojection of q).

Let an integer $n \geq 1$ be given. Note that $a^{1/n} \sim a$. Hence, by hypothesis $a^{1/n} \preceq b$. Hence, by the remarks at the end of the introduction, for every $\epsilon > 0$, there exists a positive element $b_\epsilon \in Her(b)$ such that

$$(a^{1/n} - \epsilon)_+ \approx b_\epsilon$$

Since b_ϵ necessarily has norm less than or equal to one, $b_\epsilon \leq q$. Hence, for every $\epsilon > 0$, for every normal tracial state τ on $\mathcal{M}$,

$$\tau((a^{1/n} - \epsilon)_+) = \tau(b_\epsilon) \leq \tau(q)$$

But $\tau((a^{1/n} - \epsilon)_+) \geq \tau(a^{1/n}) - \epsilon$. Hence, for every $\epsilon > 0$, for every normal tracial state τ on $\mathcal{M}$,

$$\tau(a^{1/n}) - \epsilon \leq \tau(q)$$

Since $\epsilon > 0$ is arbitrary, we have that for every normal tracial state τ on $\mathcal{M}$,

$$\tau(a^{1/n}) \leq \tau(q)$$

Since $n \geq 1$ is arbitrary, we have that for every positive integer $n \geq 1$, for every normal tracial state τ on $\mathcal{M}$,

$$\tau(a^{1/n}) \leq \tau(q)$$

But since τ is normal, $\tau(p) = limsup_{n\rightarrow\infty}\tau(a^{1/n})$. Hence for every normal tracial state τ on $\mathcal{M}$,

$$\tau(p) \leq \tau(q)$$

Hence, by [33] Theorem 8.2.8 and 8.4.3,

$$p \preceq q$$

i.e., p is Murray-von Neumman equivalent to a subprojection of q, as required. □

The next argument is similar to arguments found in [17] Theorem 3, [24], [29] and many other places (especially in classification theory). Before proceeding with the proof, we fix a notation. For a set X and for a subset $E \subseteq X$, we let χ_E denote the characteristic function for E, in other words, χ_E is the function on X such that

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X - E \end{cases}$$

Lemma 3.2. *Let $\mathcal{M}$ be a finite von Neumann algebra. Let X be a compact metric space, and let $\phi, \psi : C(X) \rightarrow \mathcal{M}$ be two unital injective $*$-homomorphisms such that for every positive element $a \in C(X)$, the support projections of $\phi(a)$ and $\psi(a)$ are Murray-von Neumann equivalent in $\mathcal{M}$.*
Then for every $\epsilon > 0$, for every finite subset $\mathcal{F} \subseteq C(X)$, there exists a unitary $u \in \mathcal{M}$ such that

$$\|\phi(f) - u\psi(f)u^*\| < \epsilon$$

for every $f \in \mathcal{F}$.

Proof. Let d be the metric for the compact metric space X. Also, for each point $z \in X$, and real number $r > 0$, we let $B(z, r)$ be the open ball of radius r with centre z; i.e., $B(z, r) = \{x \in X : d(x, z) < r\}$. Finally, for a subset $E \subseteq X$, we let ∂E be the boundary of E in X.

Contracting $\epsilon > 0$ if necessary, we may assume that the each element of $\mathcal{F}$ is positive and has norm less than or equal to one. We may also assume that $\epsilon < 1$.

By [40] Theorem 3.7.7, both ϕ and ψ extend to weak*-weak*-continuous $*$-homomorphisms $\phi'', \psi'' : Borel(X) \rightarrow \mathcal{M}$ (respectively) where $Borel(X) \cong C(X)^{**}$ is the von Neumann algebra of all bounded Borel functions on X.

Let $G \subseteq X$ be an open subset. Let $g \in C(X)$ be a continuous function such that $g(x) > 0$ for $x \in G$ and $g(x) = 0$ for $x \in X - G$. Then $\chi_G = lim_{n\rightarrow\infty} g^{1/n}$ is the support projection of of g in $Borel(X)$ (where the limit converges in the weak* topology of $Borel(X) \cong C(X)^{**}$). Since ϕ'', ψ'' are weak*-weak* continuous extensions of ϕ, ψ respectively, we have that

$$\phi''(\chi_G) = lim_{n\rightarrow\infty}\phi(g)^{1/n}$$

and

$$\psi''(\chi_G) = lim_{n\rightarrow\infty}\psi(g)^{1/n}$$

where the limits converge in the weak* topology of $\mathcal{M}$. So $\phi''(\chi_G), \psi''(\chi_G)$ are the support projections (in $\mathcal{M}$) of $\phi(g), \psi(g)$ respectively. So by hypothesis, $\phi''(\chi_G) \sim \psi''(\chi_G)$. Since G is arbitrary, we have that for every open subset $G \subseteq X$, $\phi''(\chi_G)$

and $\psi''(\chi_G)$ are Murray-von Neumann equivalent in $\mathcal{M}$. Also, if $F \subseteq X$ is closed, then, since $\mathcal{M}$ is a finite von Neumann algebra and since $X - F$ is open,

$$\phi''(\chi_F) = 1 - \phi''(\chi_{X-F}) \sim 1 - \psi''(\chi_{X-F}) = \psi''(\chi_F)$$

in $\mathcal{M}$, Hence, for every closed subset $F \subseteq X$, $\phi''(\chi_F)$ and $\psi''(\chi_F)$ are Murray-von Neumann equivalent in $\mathcal{M}$. Finally, if $F \subseteq F' \subseteq X$ is an inclusion of closed subsets of X, then, since $\mathcal{M}$ is finite and by our result for closed sets,

$$\phi''(\chi_{F'-F}) = \phi''(\chi_{F'}) - \phi''(\chi_F) \sim \psi''(\chi_{F'}) - \psi''(\chi_F) = \psi''(\chi_{F'-F})$$

in $\mathcal{M}$. We denote the above statements by "(*)".

Now since $\mathcal{F} \subseteq C(X)$ is a finite set and since (X, d) is a compact metric space, let $\delta > 0$ be a strictly positive real number such that for all $f \in \mathcal{F}$, for all $x, y \in X$,

$$d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon/100$$

Contracting δ if necessary, we may assume that $\delta < \epsilon/100$. We denote the above statements by "(**)".

Since (X, d) is compact, let $\{x_1, x_2, ..., x_m\}$ be a finite set of points in X such that $X = \bigcup_{i=1}^m B(x_i, \delta/100)$. Now let $O_1, O_2, ..., O_N$ be a collection of open subsets of X such that the following hold:

i. For each i with $1 \leq i \leq N$, there exists j with $1 \leq j \leq m$ such that $O_i \subseteq B(x_j, \delta/100)$.
ii. For each i with $1 \leq i \leq N$, $\partial O_i \subseteq \bigcup_{j=1}^m \partial B(x_j, \delta/100)$
iii. Each O_i has diameter strictly less than $\delta/50$.
iv. $X = \bigcup_{i=1}^N \overline{O_i}$

We denote the above statements by "(***)".

Now by (***), let $\{E_i\}_{i=1}^N$ be a collection of subsets of X such that the following statements hold:

(a) $\{E_i\}_{i=1}^N$ is a partition of X; i.e., $X = \bigcup_{i=1}^N E_i$ and the E_is are pairwise disjoint
(b) For $1 \leq i \leq N$, the diameter of E_i is less than or equal to $\delta/50$.
(c) For $1 \leq i \leq N$, E_i has the form $E_i = O_i \cup E_i'$ where $E_i' \subseteq \partial O_i$. (Note that the union is disjoint.)
(d) For $1 \leq i \leq N$, E_i' has the form $E_i' = F_i' - F_i$ where $F_i \subseteq F_i'$ are both closed subsets of ∂O_i.

We denote the above statements by "(****)".

Note that by by (****) statement (c), for $1 \leq i \leq N$, χ_{O_i} and $\chi_{E_i'}$ are orthogonal (since O_i and E_i' are disjoint). From this, (*) and (****), we have that for $1 \leq i \leq N$,

$$\begin{aligned} & \phi''(\chi_{E_i}) \\ = \ & \phi''(\chi_{O_i \cup E_i'}) \\ = \ & \phi''(\chi_{O_i} + \chi_{E_i'}) \\ = \ & \phi''(\chi_{O_i}) + \phi''(\chi_{F_i'-F_i}) \\ \sim \ & \psi''(\chi_{O_i}) + \psi''(\chi_{F_i'-F_i}) \\ = \ & \psi''(\chi_{O_i} + \chi_{E_i'}) \\ = \ & \psi''(\chi_{O_i \cup E_i'}) \\ = \ & \psi''(\chi_{E_i}) \end{aligned}$$

Hence, for each i, let $v_i \in \mathcal{M}$ be a partial isometry with initial projection $\phi''(\chi_{E_i})$ and range projection $\psi''(\chi_{E_i})$. Note that by (****) statement (a), we have that the $\phi''(\chi_{E_i})$s are pairwise disjoint and that the $\psi''(\chi_{E_i})$s are pairwise disjoint. Hence, from this and (****) statement (a), the element $u \in \mathcal{M}$ given by

$$u =_{df} v_1 + v_2 + ... + v_N$$

is a unitary in $\mathcal{M}$.

Finally, for $1 \leq i \leq N$, let $x_i \in E_i$ be a point. Then by (****) statement (b) and by (**), we have that for every $f \in \mathcal{F}$, f is within $\epsilon/100$ of $\sum_{i=1}^N f(x_i)\chi_{E_i}$. From this and the definition of u, we have that for all $f \in \mathcal{F}$,

$$\begin{aligned}
& \|\phi(f) - u^*\psi(f)u\| \\
= \ & \|\phi''(f) - u^*\psi''(f)u\| \\
\leq \ & \|\phi''(f) - \phi''(\sum_{i=1}^N f(x_i)\chi_{E_i})\| + \|\sum_{i=1}^N f(x_i)(\phi''(\chi_{E_i}) - u^*\psi''(\chi_{E_i})u)\| \\
& + \|u^*\psi''(\sum_{i=1}^N f(x_i)\chi_{E_i})u - u^*\psi''(f)u\| \\
< \ & \epsilon/100 + \|\sum_{i=1}^N f(x_i)(\phi''(\chi_{E_i}) - u^*\psi''(\chi_{E_i})u)\| + \epsilon/100 \\
= \ & \epsilon/50 + \|\sum_{i=1}^N f(x_i)(\phi''(\chi_{E_i}) - u^*\psi''(\chi_{E_i})u)\| \\
= \ & \epsilon/50 + max_{1 \leq i \leq N}|f(x_i)|\|\phi''(\chi_{E_i}) - u^*\psi''(\chi_{E_i})u)\| \\
= \ & \epsilon/50 + 0 \\
< \ & \epsilon
\end{aligned}$$

as required. □

From Lemma 3.2, we obtain the following corollary:

Corollary 3.3. *Let $\mathcal{M}$ be a finite von Neumann algebra, and let X be a compact metric space. Let $\phi, \psi : C(X) \to \mathcal{M}$ be two unital injective $*$-homomorphisms. Then the following are true:*

1. *If $Cu(\phi) = Cu(\psi)$ then ϕ and ψ are approximately unitarily equivalent.*
2. *If $\tau(\phi(a)) = \tau(\psi(a))$ for every positive element $a \in C(X)$ and every normal tracial state τ on $\mathcal{M}$, then ϕ and ψ are approximately unitarily equivalent.*

Proof. (1) follows from Lemma 3.1 and Lemma 3.2.

We now prove (2). Let $a \in C(X)$ be an arbitrary positive element. Let $p, q \in \mathcal{M}$ be the support projections of $\phi(a)$ and $\psi(a)$ respectively. By hypothesis, we have that for all $n \geq 1$,

$$\tau(\phi(a)^{1/n}) = \tau(\phi(a^{1/n})) = \tau(\psi(a^{1/n})) = \tau(\psi(a)^{1/n})$$

for every normal tracial state τ on $\mathcal{M}$. But $\phi(a)^{1/n} \to p$ and $\psi(a)^{1/n} \to q$ in the strong operator topology as $n \to \infty$. Hence,

$$\tau(p) = \tau(q)$$

for every normal tracial state τ on $\mathcal{M}$. Hence, by [33] Theorems 8.2.8 and 8.4.3, $p \sim q$. But $a \in C(X)$ was arbitrary. Hence, we have shown that for every positive element $a \in C(X)$, $\phi(a)$ and $\psi(a)$ have Murray-von Neumann equivalent support projections. By Lemma 3.2, this implies that ϕ and ψ are approximately unitarily equivalent, as required. □

The next proof use ideas from [17] Lemma 3.

Lemma 3.4. *Let $\mathcal{M}$ be finite von Neumann algebra. Let $\mathcal{A}$ be a unital separable C^*-algebra, and let $\phi, \psi : \mathcal{A} \to \mathcal{M}$ be two unital injective $*$-homomorphisms. Then $Cu(\phi) = Cu(\psi)$ if and only if $\tau(\phi(a)) = \tau(\psi(a))$ for every positive element $a \in \mathcal{A}$ and every normal tracial state τ on $\mathcal{M}$.*

Proof. We first prove the only if direction. Suppose that $Cu(\phi) = Cu(\psi)$. Let $a \in \mathcal{A}$ be an arbitrary positive element. Let $\mathcal{C} =_{df} C^*(a, 1_{\mathcal{A}})$ be the commutative C^*-subalgebra of $\mathcal{A}$ that is generated by a and $1_{\mathcal{A}}$. Since $\mathcal{C}$ is unital and separable, the spectrum of $\mathcal{C}$ is a compact metric space. Then $\phi|_{\mathcal{C}}, \psi|_{\mathcal{C}} : \mathcal{C} \to \mathcal{M}$ are two unital injective $*$-homomorphisms. Also, since $Cu(\phi) = Cu(\psi)$, we must have that $Cu(\phi|_{\mathcal{C}}) = Cu(\psi|_{\mathcal{C}})$. Hence, by Corollary 3.3, $\phi|_{\mathcal{C}}$ and $\psi|_{\mathcal{C}}$ are approximately unitarily equivalent. Hence, let $\{u_n\}_{n=1}^{\infty}$ be a sequence of unitaries in $\mathcal{M}$ that witnesses this. Therefore, we have that

$$u_n \phi(a)(u_n)^* \to \psi(a)$$

in norm of $\mathcal{M}$ as $n \to \infty$. Hence, for every normal tracial state τ on $\mathcal{M}$, we must have that

$$\tau(u_n \phi(a)(u_n)^*) \to \tau(\psi(a))$$

as $n \to \infty$. But for all $n \geq 1$, $\tau(u_n \phi(a)(u_n)^*) = \tau(\phi(a))$. Hence, for every normal tracial state τ on $\mathcal{M}$, $\tau(\phi(a)) = \tau(\psi(a))$. Since $a \in \mathcal{A}$ was an arbitrary positive element, this proves the only if direction.

We now prove the only if direction. Suppose that $\tau(\phi(a)) = \tau(\psi(a))$ for every positive element $a \in \mathcal{A}$ and every normal tracial state τ on $\mathcal{M}$. Note that this implies that for every positive integer $n \geq 1$, $\tilde{\tau}((\phi \otimes 1_{\mathbb{M}_n})(\tilde{a})) = \tilde{\tau}((\psi \otimes 1_{\mathbb{M}_n})(\tilde{a}))$ for every positive element $\tilde{a} \in \mathbb{M}_n(\mathcal{A}) \cong \mathcal{A} \otimes \mathbb{M}_n$ and for every normal tracial state $\tilde{\tau}$ on $\mathbb{M}_n(\mathcal{M}) \cong \mathcal{M} \otimes \mathbb{M}_n$. Hence, it suffices to show that for every positive element $a \in \mathcal{A}$, $\phi(a) \sim \psi(a)$ (i.e., $\phi(a)$ and $\psi(a)$ are Cuntz equivalent).

Let $a \in \mathcal{A}$ be a positive element. Let $\mathcal{C} =_{df} C^*(a, 1_{\mathcal{A}})$ be the unital commutative C^*-subalgebra of $\mathcal{A}$ that is generated by a and $1_{\mathcal{A}}$. Since $\mathcal{C}$ is unital and separable, the spectrum of $\mathcal{C}$ is a compact metric space. Moreover, $\phi|_{\mathcal{C}}, \psi|_{\mathcal{C}} : \mathcal{C} \to \mathcal{M}$ are two unital injective $*$-homomorphisms. Since $\tau \circ \phi = \tau \circ \psi$ for every normal tracial state τ on $\mathcal{M}$, we must have that $\tau \circ (\phi|_{\mathcal{C}}) = \tau \circ (\psi|_{\mathcal{C}})$ for every normal tracial state τ on $\mathcal{M}$. Hence, by Corollary 3.3, $\phi|_{\mathcal{C}}$ and $\psi|_{\mathcal{C}}$ are approximately unitarily equivalent. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of unitaries in $\mathcal{M}$ that witnesses this. Hence, we must have that

$$u_n \phi(a)(u_n)^* \to \psi(a)$$

and

$$(u_n)^* \psi(a) u_n \to \phi(a)$$

in norm as $n \to \infty$. Hence, $\psi(a) \preceq \phi(a)$ and $\phi(a) \preceq \psi(a)$. Hence, $\psi(a)$ and $\phi(a)$ are Cuntz equivalent in $\mathcal{M}$, as required. □

We require the following standard result, which has a short proof:

Lemma 3.5. *Let $\mathcal{B}$ be a finite-dimensional C^*-algebra.*

(1) *Let $\mathcal{C}$ be a unital C^*-algebra. Suppose that $\phi, \psi : \mathcal{B} \rightarrow \mathcal{C}$ are two unital $*$-homomorphisms such that $Cu(\phi) = Cu(\psi)$.*
Then ϕ and ψ are unitarily equivalent.
(2) *Let $\mathcal{M}$ be a finite von Neumann algebra. Suppose that $\phi, \psi : \mathcal{B} \rightarrow \mathcal{M}$ are two $*$-homomorphisms (not necessarily unital) such that $\tau \circ \phi = \tau \circ \psi$, for every normal tracial state τ on $\mathcal{M}$.*
Then ϕ and ψ are unitarily equivalent.

The next result is similar to [17] Theorem 5. We note that [17] Theorem 5 is stated incorrectly - their proof assumes the additional assumption that the von Neumann algebra is finite. Also, many steps are skipped. We write out the proof for the convenience of the reader.

Lemma 3.6. *Let $\mathcal{M}$ be a finite von Neumann algebra. Let $\mathcal{A}$ be a unital simple separable nuclear C^*-algebra. Suppose that $\phi, \psi : \mathcal{A} \rightarrow \mathcal{M}$ are unital $*$-homomorphisms such that*

$$Cu(\phi) = Cu(\psi)$$

Then ϕ and ψ are weak approximately unitarily equivalent.*

Proof. For each normal tracial state τ on $\mathcal{M}$, let $\|.\|_{2,\tau}$ be the seminorm on $\mathcal{M}$ given by $\|x\|_{2,\tau} =_{df} \tau(x^*x)^{1/2}$ for all $x \in \mathcal{M}$. By [33] 8.7.3(iii), the family of seminorms $\{\|.\|_{2,\tau} : \tau$ is a normal tracial state on $\mathcal{M}\}$ induce the strong operator topology on bounded subsets of $\mathcal{M}$. From this and by symmetry, to prove Lemma 3.6, it suffices to prove the following:

Let $\epsilon > 0$ and let $\mathcal{F} \subseteq \mathcal{A}$ be a finite subset consisting of positive elements with norm less than or equal to one. Let $\{\tau_1, \tau_2, ..., \tau_k\}$ be a finite set of normal tracial states on $\mathcal{M}$. We want to show the following: There exists a unitary $u \in \mathcal{M}$ such that for all $a \in \mathcal{F}$ and for $1 \leq i \leq k$,

$$\|\phi(a) - u\psi(a)u^*\|_{2,\tau_i} < \epsilon$$

We denote the above statement by "(*)".

Now by [40] Theorem 3.7.7, both ϕ and ψ extend to weak*-weak*-continuous $*$-homomorphisms $\phi'', \psi'' : \mathcal{A}^{**} \rightarrow \mathcal{M}$ respectively (where $\mathcal{A}^{**}$ is the second dual (or enveloping) von Neumann algebra of $\mathcal{A}$). By [33] Proposition 7.1.15, both ϕ'' and ψ'' are strong operator-strong operator continuous on bounded subsets of $\mathcal{A}^{**}$. We denote the above statements by "(**)".

Since $\mathcal{A}$ is a nuclear C^*-algebra, it follows by the Choi-Effros Theorem [8], that $\mathcal{A}^{**}$ is an injective von Neumann algebra. Hence, by [21] and since ϕ'', ψ'' are both strong operator-strong operator continuous, there is finite dimensional C^*-subalgebra $\mathcal{D}$ of $\mathcal{A}^{**}$ where for every $a \in \mathcal{F}$, there exists $h(a) \in \mathcal{D}$ such that for $1 \leq i \leq k$,

$$\|\phi''(a) - \phi''(h(a))\|_{2,\tau_i} < \epsilon/100$$

and

$$\|\psi''(a) - \psi''(h(a))\|_{2,\tau_i} < \epsilon/100$$

We denote the above statement by "(***)".

We have two (not necessarily unital) $*$-homomorphisms $\phi''|_{\mathcal{D}}, \psi''|_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{M}$. Since $Cu(\phi) = Cu(\psi)$, we have, by Lemma 3.4, that $\tau \circ \phi = \tau \circ \psi$ for all normal tracial states τ on $\mathcal{M}$. Since ϕ'' and ψ'' are weak*-weak* continuous, this implies that $\tau \circ \phi'' = \tau \circ \psi''$ for all normal tracial states τ on $\mathcal{M}$. Hence, $\tau \circ (\phi''|_{\mathcal{D}}) = \tau \circ (\psi''|_{\mathcal{D}})$ for all normal tracial states τ on $\mathcal{M}$. Hence, by Lemma 3.5, let $u \in \mathcal{M}$ be a unitary such that $\phi''(b) = u\psi''(b)u^*$ for every $b \in \mathcal{D}$. We denote the above statement by "(****)".

From our assumptions on $\mathcal{F}$, (***) and (****), we have that for every $a \in \mathcal{F}$, for $1 \leq i \leq k$,

$$\begin{aligned}
& \|\phi(a) - u\psi(a)u^*\|_{2,\tau_i} \\
= \; & \|\phi''(a) - u\psi''(a)u^*\|_{2,\tau_i} \\
\leq \; & \|\phi''(a) - \phi''(h(a))\|_{2,\tau_i} + \|\phi''(h(a)) - u\psi''(h(a))u^*\|_{2,\tau_i} + \\
& \|u\psi''(h(a))u^* - u\psi''(a)u^*\|_{2,\tau_i} \\
< \; & \epsilon/100 + 0 + \|u\psi''(h(a))u^* - u\psi''(a)u^*\|_{2,\tau_i} \\
= \; & \epsilon/100 + \tau_i((u\psi''(h(a))u^* - u\psi''(a)u^*)^*(u\psi''(h(a))u^* - u\psi''(a)u^*))^{1/2} \\
= \; & \epsilon/100 + \tau_i(u(\psi''(h(a)) - \psi''(a))^2 u^*)^{1/2} \\
= \; & \epsilon/100 + \tau_i((\psi''(h(a)) - \psi''(a))^2)^{1/2} \\
= \; & \epsilon/100 + \|\psi''(h(a)) - \psi''(a)\|_{2,\tau_i} \\
< \; & \epsilon/100 + \epsilon/100 \\
< \; & \epsilon
\end{aligned}$$

as required. □

4. Nuclearity and uniqueness

Consider the Cuntz algebra O_2. (So O_2 is the (unique simple, unital, separable, purely infinite) C^*-algebra generated by isometries S, T such that $1 = SS^* + TT^*$.) To obtain our characterization of nuclearity, we need the following classification theory result whose (short) proof we provide for the convenience of the reader.

Lemma 4.1. *O_2 can be expressed as an increasing union (or inductive limit) of the following form:*

$$O_2 = \overline{\bigcup_{n \geq 1} \bigotimes^n O_2}$$

where for all $n \geq 1$, $\bigotimes^n O_2$ is the n times tensor product of O_2 with itself (i.e., $\bigotimes^n O_2 = O_2 \otimes O_2 \otimes ... \otimes O_2$ where, on the right hand side, O_2 is repeated n times), and where the (unital injective) connecting maps $\otimes^n O_2 \to \otimes^{n+1} O_2$ have the form $a \mapsto a \otimes 1_{O_2}$.

Proof. By [41] and [34], we have that for $n \geq 1$, $\bigotimes^n O_2 \cong O_2$. Hence, by continuity of the K-groups, $\overline{\bigcup_{n \geq 1} \bigoplus^n O_2}$ has trivial K-theory. But $\overline{\bigcup_{n \geq 1} \bigoplus^n O_2}$ must be unital, simple, separable, nuclear, purely infinite and satisfies the universal coefficient theorem, since the building blocks have these properties. Hence, by [41] and [34], $\overline{\bigcup_{n \geq 1} \bigoplus^n O_2} \cong O_2$ as required. □

Definition 4.1. *Let $\mathcal{A}$ be a simple unital separable C^*-algebra. We say that $\mathcal{A}$ has the* weak uniqueness property *if the following is true:*

For every unital separable C^*-algebra $\mathcal{C}$, for all unital injective $*$-homomorphisms $\phi, \psi : \mathcal{A} \to \mathcal{C}$ such that $Cu(\phi) = Cu(\psi)$, ϕ and ψ are weakly approximately unitarily equivalent.

We also need the following lemma concerning moving up matrices for weak uniqueness:

Lemma 4.2. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra. If $\mathcal{A}$ has the weak uniqueness property then for every integer $n \geq 1$, $\mathbb{M}_n(\mathcal{A})$ has the weak uniqueness property.*

Proof. Suppose that $\mathcal{C}$ is a unital separable C^*-algebra and $\phi, \psi : \mathbb{M}_n(\mathcal{A}) \to \mathcal{C}$ are unital injective $*$-homomorphisms such that $Cu(\phi) = Cu(\psi)$. We want to show that ϕ and ψ are weakly approximately unitarily equivalent. To do this, it suffices to prove the following:

Let $\epsilon > 0$ be given. Let $\mathcal{F} \subseteq \mathbb{M}_n \otimes \mathcal{A} \cong \mathbb{M}_n(\mathcal{A})$ be a finite subset consisting of positive elements with norm less than or equal to one, and let $\rho_1, \rho_2, ..., \rho_k$ be a finite list of states on $\mathcal{C}$ (so $\rho_i \in \mathcal{C}^*$ for all i). Then there exists a unitary $u \in \mathcal{C}$ such that

$$\|\phi(b) - u\psi(b)u^*\|_{\rho_i} < \epsilon$$

for $1 \leq i \leq k$ and for $b \in \mathcal{F}$.
We denote the above statement by "(*)".

Let $\{e_{i,j}\}_{1 \leq i,j \leq n}$ be the standard system of matrix units for $\mathbb{M}_n$. Then $\{1_{\mathcal{A}} \otimes e_{i,j}\}$ is the standard system of matrix units for $1_{\mathcal{A}} \otimes \mathbb{M}_n \subseteq \mathcal{A} \otimes \mathbb{M}_n \cong \mathbb{M}_n(\mathcal{A})$. (We will henceforth identify $\mathcal{A} \otimes \mathbb{M}_n$ with $\mathbb{M}_n(\mathcal{A})$.) Since $Cu(\phi) = Cu(\psi)$, we must have that $Cu(\phi|_{1_{\mathcal{A}} \otimes \mathbb{M}_n}) = Cu(\psi|_{1_{\mathcal{A}} \otimes \mathbb{M}_n})$. Hence, by Lemma 3.5, let $w \in \mathcal{C}$ be a unitary such that $\phi(1_{\mathcal{A}} \otimes e_{i,j}) = Ad(w)\psi(1_{\mathcal{A}} \otimes e_{i,j}) =_{df} w\psi(1_{\mathcal{A}} \otimes e_{i,j})w^*$ for $1 \leq i, j \leq n$.

Let $\phi', \psi' : \mathcal{A} \otimes e_{1,1} \to \phi(1_{\mathcal{A}} \otimes e_{1,1})\mathcal{C}\phi(1_{\mathcal{A}} \otimes e_{1,1})$ be two unital injective $*$-homomorphisms which are given by $\phi' =_{df} \phi|_{\mathcal{A} \otimes e_{1,1}}$ and $\psi' =_{df} Ad(w)\psi|_{\mathcal{A} \otimes e_{1,1}}$.

Note that for $1 \leq i, j \leq n$, for $a \in \mathcal{A}$, $\phi(a \otimes e_{i,j}) = \phi(1_{\mathcal{A}} \otimes e_{i,1})\phi'(a \otimes e_{1,1})\phi(1_{\mathcal{A}} \otimes e_{1,j})$ and $(Ad(w)\psi)(a \otimes e_{i,j}) = \phi(1_{\mathcal{A}} \otimes e_{i,1})\psi'(a \otimes e_{1,1})\phi(1_{\mathcal{A}} \otimes e_{1,j})$. We denote the above statements by "(**)".

Note also that since $Cu(\phi) = Cu(\psi)$, we must have that $Cu(\phi') = Cu(\psi')$. Since $\mathcal{A} \cong \mathcal{A} \otimes e_{1,1}$ and since $\mathcal{A}$ has the weak uniqueness property, ϕ' and ψ' are weakly approximately unitarily equivalent. Let $\{w_\alpha\}_{\alpha \in I}$ be a net of unitaries in $\phi(1_{\mathcal{A}} \otimes e_{1,1})\mathcal{C}\phi(1_{\mathcal{A}} \otimes e_{1,1})$ such that for all $c \in \mathcal{A} \otimes e_{1,1}$, $Ad(w_\alpha)\psi'(c) \to \phi'(c)$ in the relative weak topology for $\phi(1_{\mathcal{A}} \otimes e_{1,1})\mathcal{C}\phi(1_{\mathcal{A}} \otimes e_{1,1})$. Hence, by [40] Proposition 3.1.6, for all $c \in \mathcal{A} \otimes e_{1,1}$, $Ad(w_\alpha)\psi'(c) \to \phi'(c)$ in the relative weak topology for $\mathcal{C}$. We denote the above statement by "(***)".

Now suppose that $\mathcal{F}$ has the form $\mathcal{F} = \{b_1, b_2, ..., b_L\}$ where for $1 \leq l \leq L$, b_l has the form

$$b_l = \sum_{1 \leq i,j \leq n} a_{l,i,j} \otimes e_{i,j}$$

where each $a_{l,i,j} \in \mathcal{A}$. Note that since each element of $\mathcal{F}$ has norm less than or equal to one, $\|a_{l,i,j}\| \leq 1$ for all l, i, j. We denote the above statements by "(****)".

By (***) and (****), choose $\alpha_0 \in I$ such that for all $\alpha \geq \alpha_0$, $1 \leq l \leq L$ and $1 \leq i, j \leq n$, for $1 \leq s \leq k$, and for $1 \leq t, m \leq n$,

$$\|Ad(w_\alpha)\psi'(a_{l,i,j} \otimes e_{1,1}) - \phi'(a_{l,i,j} \otimes e_{1,1})\|_{\rho_{s,t,m}} < \epsilon/(1000n^2)$$

where $\rho_{s,t,m} \in \mathcal{C}^*$ is such that $\rho_{s,t,m}(c) =_{df} \rho(\phi(1 \otimes e_{t,1})c\phi(1 \otimes e_{1,m}))$ for $c \in \mathcal{C}$. We denote the above statements by "(*****)".

Now let $u \in \mathcal{C}$ be the unitary that is given by

$$u =_{df} w'w$$

where

$$w' =_{df} \sum_{t=1}^{n} \phi(1 \otimes e_{t,1}) w_{\alpha_0} \phi(1 \otimes e_{1,t})$$

By (**), (****), (*****) and the definition of u, for $1 \leq l \leq L$ and $1 \leq s \leq k$,

$$\begin{aligned}
& \|u\psi(b_l)u^* - \phi(b_l)\|_{\rho_s} \\
= \;& \|w'w\psi(b_l)w^*(w')^* - \phi(b_l)\|_{\rho_s} \\
= \;& \|\sum_{1 \leq i,j \leq n} (w'w\psi(a_{l,i,j} \otimes e_{i,j})w^*(w')^* - \phi(a_{l,i,j} \otimes e_{i,j}))\|_{\rho_s} \\
= \;& \|\sum_{1 \leq i,j \leq n} (w'\phi(1 \otimes e_{i,1})\psi'(a_{l,i,j} \otimes e_{1,1})\phi(1 \otimes e_{1,j})(w')^* \\
& -\phi(1 \otimes e_{i,1})\phi'(a_{l,i,j} \otimes e_{1,1})\phi(1 \otimes e_{1,j}))\|_{\rho_s} \\
\leq \;& \|\sum_{1 \leq i,j \leq n} (\phi(1 \otimes e_{i,1})w_{\alpha_0}\phi(1 \otimes e_{1,1})\psi'(a_{l,i,j} \otimes e_{1,1})\phi(1 \otimes e_{1,1})(w_{\alpha_0})^*\phi(1 \otimes e_{1,j}) \\
& -\phi(1 \otimes e_{i,1})\phi'(a_{l,i,j} \otimes e_{1,1})\phi(1 \otimes e_{1,j}))\|_{\rho_s} \\
\leq \;& \sum_{1 \leq i,j \leq n} \|w_{\alpha_0}\psi'(a_{l,i,j} \otimes e_{1,1})(w'_{\alpha_0})^* - \phi'(a_{l,i,j} \otimes e_{1,1})\|_{\rho_{s,i,j}} \\
< \;& \sum_{1 \leq i,j \leq n} \epsilon/(1000n^2) \\
= \;& \epsilon/1000 \\
< \;& \epsilon
\end{aligned}$$

as required. By (*), we have completed the proof of Lemma 4.2. □

For a von Neumman algebra $\mathcal{M}$, recall that the σ-strong* topology on $\mathcal{M}$ is the topology on $\mathcal{M}$ generated by all the seminorms $\|.\|^{\sharp}_{2,\rho}$, where ρ ranges over all normal states in $\mathcal{M}_*$ (see [47] Definition 2.3). We next need the following theorem, which is due to Elliott and Woods (see [26]).

Lemma 4.3. *Let $\mathcal{M}$ be a countably generated, properly infinite von Neumann algebra. Suppose that $\mathcal{M}$ can be locally σ-strong* approximated by finite dimensional unital C^*-subalgebras; more precisely, suppose that we have the following:*

Suppose that for every $\epsilon > 0$, for every finite subset $\mathcal{F} \subseteq \mathcal{M}$, for every finite collection $\rho_1, \rho_2, ..., \rho_k$ of normal states in $\mathcal{M}_*$, there exists a finite dimensional unital C^*-subalgebra $\mathcal{E} \subseteq \mathcal{M}$, and for each $b \in \mathcal{F}$ there is $h(b) \in \mathcal{E}$ such that for $1 \leq i \leq k$,

$$\|b - h(b)\|^{\sharp}_{2,\rho_i} < \epsilon$$

Then $\mathcal{M}$ is an injective von Neumann algebra.

Before proving our characterization theorem, we remind the reader that $\otimes$ is the minimal tensor product for C^*-algebras, $\odot$ is the algebraic tensor product for C^*-algebras, and $\overline{\otimes}$ is the von Neumann algebraic tensor product for von Neumann algebras.

Also, in order to avoid confusion for experts in von Neumann algebras, we make the following clear: All throughout the following proof, for a C^*-algebra $\mathcal{C}$, $\mathcal{C}'$ is *not* notation for the commutant of $\mathcal{C}$ on some Hilbert space or inside some bigger C^*-algebra. Similar for notation like $\mathcal{C}''$ etc.

Theorem 4.4. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra. Then the following are equivalent:*

(1) *$\mathcal{A}$ is nuclear*
(2) *$\mathcal{A}$ has the weak uniqueness property. In other words, the following is true:*
   For every unital separable C^*-algebra $\mathcal{C}$, for all unital injective $*$-homomorphisms $\phi, \psi : \mathcal{A} \to \mathcal{C}$ such that $Cu(\phi) = Cu(\psi)$, ϕ and ψ are weakly approximately unitarily equivalent.

Proof. If $\mathcal{A}$ is type I then $\mathcal{A}$ is a full matrix algebra (i.e., a simple finite dimensional C^*-algebra), and both statements (1) and (2) are clear. So we may assume that $\mathcal{A}$ is not type I.

We first prove that (1) implies (2). It suffices to prove that for every $\epsilon > 0$, for every finite subset $\mathcal{F} \subseteq \mathcal{A}$ of positive elements with norm less than or equal to one, for every finite set $\{\rho_1, \rho_2, ..., \rho_k\}$ of states in $\mathcal{C}^*$, there exists a unitary $u \in \mathcal{C}$ such that

$$\|\phi(b) - u\psi(b)u^*\|_{\rho_i} < \epsilon$$

for all $b \in \mathcal{F}$ and for $1 \leq i \leq k$. We denote the above statement by "(*)".

The natural $*$-embedding $i : \mathcal{C} \to \mathcal{C}^{**}$ induces natural $*$-embeddings $i \circ \phi, i \circ \psi : \mathcal{A} \to \mathcal{C}^{**}$.

By [47] Theorem V.1.19, the (second dual, or enveloping) von Neumann algebra $\mathcal{C}^{**}$ can be decomposed as a direct sum:

$$\mathcal{C}^{**} \cong \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$$

where $\mathcal{M}_1$ is a properly infinite von Neumann algebra, $\mathcal{M}_2$ is a finite type II_1 von Neumann algebra, and $\mathcal{M}_3$ is a finite type I von Neumann algebra (hence, by [47] Theorem V.1.27, $\mathcal{M}_3$ is a direct product of von Neumann algebras of the form $\mathcal{N} \otimes \mathbb{M}_n$ where $n \geq 1$ is a positive integer and $\mathcal{N}$ is a commutative von Neumann algebra). For $1 \leq j \leq 3$, let $\pi_j : \mathcal{C}^{**} \to \mathcal{M}_j$ be the natural projection map. For $1 \leq j \leq 3$, let $\phi_j =_{df} \pi_j \circ i \circ \phi$ and let $\psi_j =_{df} \pi_j \circ i \circ \psi$. So $i \circ \phi = \phi_1 \oplus \phi_2 \oplus \phi_3$ and $i \circ \psi = \psi_1 \oplus \psi_2 \oplus \psi_3$.

Now since $\mathcal{A}$ is a unital separable simple C^*-algebra which is *not* type I, and since ϕ_3, ψ_3 are both unital, we must have that $\mathcal{M}_3 = 0$ and $\phi_3 = \psi_3 = 0$. Also, by Lemma 2.6, ϕ_1 and ψ_1 are weak* approximately unitarily equivalent. Hence, let $\{w_\alpha\}_{\alpha \in I}$ be a net of unitaries in $\mathcal{M}_1$ such that

$$w_\alpha \psi_1(a)(w_\alpha)^* \to \phi_1(a)$$

in the weak*-topology (on $\mathcal{M}_1$ or $\mathcal{C}^{**}$), for all $a \in \mathcal{A}$. Also, since $Cu(\phi) = Cu(\psi)$, $Cu(i \circ \phi) = Cu(i \circ \psi)$ and $Cu(\phi_2) = Cu(\psi_2)$. Hence, by Lemma 3.6, ϕ_2 and ψ_2 are weak* approximately unitarily equivalent. Hence, let $\{w'_\beta\}_{\beta \in J}$ be a net of unitaries in $\mathcal{M}_2$ such that

$$w'_\beta \psi_2(a)(w'_\beta)^* \to \phi_2(a)$$

in the weak* topology (on $\mathcal{M}_2$ or $\mathcal{C}^{**}$), for all $a \in \mathcal{A}$.

Consider the directed set $I \times J$, where the order is given by $(\alpha', \beta') \leq (\alpha, \beta)$ if and only if $\alpha' \leq \alpha$ and $\beta' \leq \beta$, for all $(\alpha, \beta), (\alpha', \beta') \in I \times J$. Let $\{u_{(\alpha,\beta)}\}_{(\alpha,\beta) \in I \times J}$ be the net of unitaries in $\mathcal{M}$ that is given by

$$u_{(\alpha,\beta)} =_{df} w_\alpha \oplus w'_\beta$$

for all $(\alpha, \beta) \in I \times J$. Then it follows that

$$u_{(\alpha,\beta)}(i \circ \psi(a))(u_{(\alpha,\beta)})^* \to i \circ \phi(a)$$

in the weak* topology (on $\mathcal{C}^{**}$) for all $a \in \mathcal{A}$. Hence, choose $(\alpha_0, \beta_0) \in I \times J$ such that for all $(\alpha, \beta) \in I \times J$ such that $(\alpha, \beta) \geq (\alpha_0, \beta_0)$, for all $b \in \mathcal{F}$ and for $1 \leq i \leq k$,

$$\|u_{\alpha,\beta}\psi(a)(u_{\alpha,\beta})^* - \phi(a)\|_{\rho_i} < \epsilon/100$$

We denote the above statement by "(**)".

By [47] Lemma II.2.5 and Theorem II.4.11, let $u \in \mathcal{C}$ be a unitary such that for $1 \leq i \leq k$

$$\|(u_{\alpha_0,\beta_0} - u)^*(u_{\alpha_0,\beta_0} - u)\|_{\rho_i} + \|(u_{\alpha_0,\beta_0} - u)(u_{\alpha_0,\beta_0} - u)^*\|_{\rho_i} < (\epsilon/100)^2$$

We denote the above statement by "(***)".

From (**), (***), the requirements on $\mathcal{F}$ and the ρ_i and the Cauchy-Schwarz inequality (see [32] Proposition 4.3.1), we have that that for $b \in \mathcal{F}$, for $1 \leq i \leq k$,

$$\begin{aligned}
& \|\phi(b) - u\psi(b)u^*\|_{\rho_i} \\
\leq \; & \|\phi(b) - u_{\alpha_0,\beta_0}\psi(b)(u_{\alpha_0,\beta_0})^*\|_{\rho_i} + \|u_{\alpha_0,\beta_0}\psi(b)(u_{\alpha_0,\beta_0})^* - u_{\alpha_0,\beta_0}\psi(b)u^*\|_{\rho_i} + \\
& \|u_{\alpha_0,\beta_0}\psi(b)u^* - u\psi(b)u^*\|_{\rho_i} \\
< \; & \epsilon/100 + |\rho_i(u_{\alpha_0,\beta_0}\psi(b)(u_{\alpha_0,\beta_0} - u)^*)| + |\rho_i((u_{\alpha_0,\beta_0} - u)\psi(b)u^*)| \\
\leq \; & \epsilon/100 + \rho_i(u_{\alpha_0,\beta_0}\psi(b)^2(u_{\alpha_0,\beta_0})^*)^{1/2}\rho_i((u_{\alpha_0,\beta_0} - u)(u_{\alpha_0,\beta_0} - u^*))^{1/2} + \\
& \rho_i((u_{\alpha_0,\beta_0} - u)(u_{\alpha_0,\beta_0} - u)^*)^{1/2}\rho_i(u\psi(b)^2u^*)^{1/2} \\
< \; & \epsilon/100 + \rho_i(u_{\alpha_0,\beta_0}(u_{\alpha_0,\beta_0})^*)^{1/2}(\epsilon/100) + (\epsilon/100)\rho_i(uu^*)^{1/2} \\
= \; & \epsilon/100 + \rho_i(1)^{1/2}(\epsilon/100) + (\epsilon/100)\rho_i(1)^{1/2} \\
= \; & \epsilon/100 + \epsilon/100 + \epsilon/100 \\
< \; & \epsilon
\end{aligned}$$

as required. We have proven (*) and, hence, the direction (1) implies (2).

We next prove that (2) implies (1). Let $\mathcal{A}'$ be a unital C^*-algebra such that $\mathcal{A}'$ is $*$-isomorphic to the Cuntz algebra O_2 (so $\mathcal{A}' \cong O_2$) and let $\mathcal{K}$ be the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space. Since $\mathcal{A}$ is unital simple and separable and since $\mathcal{A}'$ is simple purely infinite, the Cuntz semigroup of $\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{A}'$ is trivial. (i.e., $Cu(\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{A}')$ is a two element set, consisting of the zero element and a nonzero element.)

Claim 1: $\mathcal{M} =_{df} \mathcal{A}^{**}\overline{\otimes}\mathcal{K}^{**}\overline{\otimes}(\mathcal{A}')^{**}$ is an injective von Neumann algebra.

Note that $\mathcal{M} = \mathcal{A}^{**}\overline{\otimes}\mathcal{K}^{**}\overline{\otimes}(\mathcal{A}')^{**}$ is properly infinite. Also, by [47] Proposition IV.4.13 and [40] Theorem 2.2.2, $\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{A}'$ is strongly dense in $\mathcal{M} = \mathcal{A}^{**}\overline{\otimes}\mathcal{K}^{**}\overline{\otimes}(\mathcal{A}')^{**}$. Hence, since $\mathcal{A} \otimes \mathcal{K} \otimes \mathcal{A}'$ is separable, $\mathcal{M} = \mathcal{A}^{**}\overline{\otimes}\mathcal{K}^{**}\overline{\otimes}(\mathcal{A}')^{**}$ is countably generated. Hence, we will use Lemma 4.3 to prove Claim 1.

Towards applying Lemma 4.3, let $\epsilon > 0$, and a finite subset $\mathcal{F} \subset \mathcal{M} = \mathcal{A}^{**}\overline{\otimes}\mathcal{K}^{**}\overline{\otimes}(\mathcal{A}')^{**}$ given. Let $\{\rho_1, \rho_2, ..., \rho_k\}$ be a finite set of normal states in the predual $\mathcal{M}_*$. Contracting $\epsilon > 0$ if necessary, we may assume that the elements of $\mathcal{F}$ are positive and have norm less than or equal to one. We may also assume that $\epsilon < 1/100$.

By Lemma 4.3, it suffices to prove the following:
There exists a unital C^*-subalgebra $\mathcal{D} \subseteq \mathcal{M}$ such that the following hold:

(1) $\mathcal{D}$ is a finite dimensional C^*-algebra.
(2) For every $b \in \mathcal{F}$, there exists $e(b) \in \mathcal{D}$ such that for $1 \leq i \leq k$,

$$\|b - e(b)\|^{\sharp}_{2,\rho_i} < \epsilon$$

We denote the above statements by "(+)".

Now let $\mathcal{E} \subseteq \mathcal{K}^{**} \cong \mathbb{B}(\mathcal{H})$ be a unital C^*-subalgebra such that $\mathcal{E}$ is a UHF algebra and $\mathcal{E}$ is σ-strong* dense in $\mathcal{K}^{**}$. Hence, by [47] Proposition IV.4.13, $\mathcal{A} \otimes \mathcal{E} \otimes \mathcal{A}'$ is σ-strong* dense in $\mathcal{M}$ (see also [47] Theorem II.4.8, Lemma II.2.5 and [16] Theorem I.7.1). Hence, from this and the Kaplansky Density Theorem (see [16] Theorem I.7.3, and [47] Theorem II.4.8), for each $b \in \mathcal{F}$, let $e'(b) \in \mathcal{A} \otimes \mathcal{E} \otimes \mathcal{A}'$ be a positive element with norm less than or equal to one, such that for $1 \leq i \leq k$,

$$\|b - e'(b)\|^{\sharp}_{2,\rho_i} < \epsilon/2$$

Hence, by (+), it suffices to prove the following: There exists a unital C^*-subalgebra $\mathcal{D} \subseteq \mathcal{M}$ such that the following hold:

(1) $\mathcal{D}$ is a finite dimensional C^*-algebra.
(2) For every $b \in \mathcal{F}$, there exists $e(b) \in \mathcal{D}$ such that for $1 \leq i \leq k$,

$$\|e'(b) - e(b)\|^{\sharp}_{2,\rho_i} < \epsilon/2$$

We denote the above statement by "(++)".

Now by the Russo-Dye Theorem (see [16] Theorem I.8.4) and by basic properties of tensor products, $\mathcal{A} \otimes \mathcal{E} \otimes \mathcal{A}'$ is the norm-closure of the linear span of all unitaries (in $\mathcal{A} \otimes \mathcal{E} \otimes \mathcal{A}'$) of the form $v_0 \otimes v_0' \otimes v_0''$ where $v_0 \in \mathcal{A}$, $v_0' \in \mathcal{E}$, $v_0'' \in \mathcal{A}'$, and v_0, v_0', v_0'' are all unitaries. Hence, let $\delta > 0$ be a strictly positive real number with $\delta < \epsilon$, and let $\{u_j \otimes u_j' \otimes u_j'' : 1 \leq j \leq L\}$ be a finite set of elements in $\mathcal{A} \otimes \mathcal{E} \otimes \mathcal{A}'$ such that the following hold:

i. $u_j \in \mathcal{A}$, $u_j' \in \mathcal{E}$, and $u_j'' \in \mathcal{A}'$ for $1 \leq j \leq L$
ii. u_j, u_j', u_j'' are all unitaries, for $1 \leq j \leq L$
iii. Suppose that $\mathcal{D} \subseteq \mathcal{M}$ is a unital C^*-subalgebra such that for $1 \leq j \leq L$, there exists $u_j''' \in \mathcal{D}$, with norm less than or equal to one, such that for $1 \leq i \leq k$,

$$\|u_j \otimes u_j' \otimes u_j'' - u_j'''\|^{\sharp}_{2,\rho_i} < \delta$$

Then for $b \in \mathcal{F}$, there exists $e(b) \in \mathcal{D}$ such that for $1 \leq i \leq k$,

$$\|e(b) - e'(b)\|^{\sharp}_{2,\rho_i} < \epsilon/2$$

From the above and (++), it suffices to prove the following: There exists a unital C^*-subalgebra $\mathcal{D} \subseteq \mathcal{M}$ such that the following hold:

(1) $\mathcal{D}$ is a finite dimensional C^*-algebra.
(2) For $1 \leq j \leq L$, there exists a unitary $u_j''' \in \mathcal{D}$ such that for $1 \leq i \leq k$,

$$\|u_j \otimes u_j' \otimes u_j'' - u_j'''\|^{\sharp}_{2,\rho_i} < \delta$$

We denote the above statements by "(+++)".

Next, by [33] Proposition 11.2.8, if $\{c_{\alpha'}\}$, $\{d_{\alpha''}\}$ and $\{c'_{\alpha'''}\}$ are (norm) bounded nets in $\mathcal{A}^{**}$, $\mathcal{K}^{**}$, $(\mathcal{A}')^{**}$ respectively, and if $c_{\alpha'} \to c'$, $d_{\alpha''} \to d''$ and $c'_{\alpha'''} \to c''$ in the weak* topologies of $\mathcal{A}^{**}$, $\mathcal{K}^{**}$ and $(\mathcal{A}')^{**}$ respectively, then $c_{\alpha'} \otimes d_{\alpha''} \otimes c'_{\alpha'''} \to c' \otimes d'' \otimes c''$ in the weak* topology on $A^{**} \overline{\otimes} \mathcal{K}^{**} \overline{\otimes} (\mathcal{A}')^{**}$. We denote the above statements by "(++++)".

Since $\mathcal{E}$ is a UHF algebra, since the minimal tensor product norm is a subcross norm (see [53]), and since the norm topology is stronger than the σ-strong* topology on $\mathcal{M}$, let $N \geq 1$ be a positive integer, let $\mathcal{E}_1 \subseteq \mathcal{E}$ be a unital C^*-subalgebra with

$\mathcal{E}_1 \cong \mathbb{M}_N$, and for $1 \leq j \leq L$, let $v'_j \in \mathcal{E}_1$ be a unitary such that for $1 \leq j \leq L$ and $1 \leq i \leq k$,

$$\|u_j \otimes v'_j \otimes u''_j - u_j \otimes u'_j \otimes u''_j\|^\sharp_{2,\rho_i} < \delta/1000$$

We denote the above statement by "(+++++)".

Note that in the inductive limit in Lemma 4.1, each building block $\bigotimes^n O_2$ is $*$-isomorphic to O_2 (see [41] and [34]). Hence, by Lemma 4.1, since the minimal tensor product norm is subcross, and since the norm topology is stronger than the σ-strong* topology on $\mathcal{M}$, there is a unital C^*-subalgebra $\mathcal{A}'' \subseteq \mathcal{A}'$, with a tensor product decomposition $\mathcal{A}'' \cong \mathcal{A}''_1 \otimes \mathcal{A}''_2$ such that the following statements are true:

(a) $\mathcal{A}''_i \cong O_2$ for $1 \leq i \leq 2$

(b) For $1 \leq j \leq L$, there exists a unitary $v''_j \in \mathcal{A}''_2$ such that for $1 \leq i \leq k$,

$$\|u_j \otimes v'_j \otimes 1_{\mathcal{A}''_1} \otimes v''_j - u_j \otimes v'_j \otimes u''_j\|^\sharp_{2,\rho_i} < \delta/1000$$

From this and (+++++), we have that for $1 \leq j \leq L$ and $1 \leq i \leq k$,

$$\|u_j \otimes v'_j \otimes 1_{\mathcal{A}''_1} \otimes v''_j - u_j \otimes u'_j \otimes u''_j\|^\sharp_{2,\rho_i} < \delta/500$$

We denote the above statement by "(++++++)".

Note that $\mathcal{A} \otimes \mathbb{M}_N$ is a unital separable simple C^*-algebra. Now let $\phi' : \mathcal{A} \otimes \mathbb{M}_N \rightarrow \mathcal{A} \otimes \mathcal{E}_1$ be the natural identity map (where we identify $\mathbb{M}_N$ with $\mathcal{E}_1$). Also, let $\psi' : \mathcal{A} \otimes \mathbb{M}_N \rightarrow \mathbb{B}(\mathcal{H})$ be an arbitrary unital $*$-homomorphism. (ψ' is necessarily injective since $\mathcal{A} \otimes \mathbb{M}_N$ is simple.) Let $\mathcal{C}' \subseteq \mathcal{A} \otimes \mathbb{B}(\mathcal{H})$ be the separable unital C^*-subalgebra that is generated by $\mathcal{A} \otimes \mathcal{E}$ and $\mathcal{A} \otimes Image(\psi')$. Let $\mathcal{C} \subseteq \mathcal{A} \otimes \mathbb{B}(\mathcal{H}) \otimes \mathcal{A}''_1 \otimes 1_{\mathcal{A}''_2}$ be the separable unital C^*-subalgebra that is given by $\mathcal{C} =_{df} \mathcal{C}' \otimes \mathcal{A}''_1 \otimes 1_{\mathcal{A}''_2}$. Let $\phi, \psi : \mathcal{A} \otimes \mathbb{M}_N \rightarrow \mathcal{C}$ be unital injective $*$-homomorphisms that are given by $\phi =_{df} \phi' \otimes 1_{\mathcal{A}''_1} \otimes 1_{\mathcal{A}''_2}$ and $\psi =_{df} 1_{\mathcal{A}} \otimes \psi' \otimes 1_{\mathcal{A}''_1} \otimes 1_{\mathcal{A}''_2}$.

Note that since $\mathcal{A}''_1 \cong O_2$, we have that $\mathcal{C} \otimes O_2 \cong \mathcal{C}$. But by [41] and [34], $O_2 \otimes O_\infty \cong O_2$. Hence, $\mathcal{C} \otimes O_\infty \cong \mathcal{C}$. Hence, by [35] Proposition 4.5, since ϕ and ψ are both full $*$-homomorphisms, $Cu(\phi) = Cu(\psi)$. From this, the hypothesis on $\mathcal{A}$ and by Lemma 4.2, we have that ϕ and ψ are weakly approximately unitarily equivalent (with codomain $\mathcal{C}$). Hence, Let $\{u_\alpha\}_{\alpha \in I}$ be a net of unitaries in $\mathcal{C}$ such that for all $b \in \mathcal{A} \otimes \mathbb{M}_N$,

$$u_\alpha \psi(b)(u_\alpha)^* \rightarrow \phi(b)$$

in the relative weak topology on $\mathcal{C}$ (equivalently, the weak* topology on the von Neumann algebra $\mathcal{C}^{**}$). Note that by [47] Proposition IV.4.13, $\mathcal{C}$ is a (unital) C^*-subalgebra of $\mathcal{M} =_{df} \mathcal{A}^{**} \overline{\otimes} \mathcal{K}^{**} \overline{\otimes} (\mathcal{A}')^{**}$. So any element of the predual $\mathcal{M}_*$ gives a continuous linear functional on $\mathcal{C}$ (i.e., gives an element of $\mathcal{C}^*$). Hence, for all $b \in \mathcal{A} \otimes \mathbb{M}_N$,

$$u_\alpha \psi(b)(u_\alpha)^* \rightarrow \phi(b)$$

in the weak* topology on $\mathcal{M} = \mathcal{A}^{**} \overline{\otimes} \mathcal{K}^{**} \overline{\otimes} (\mathcal{A}')^{**}$. Hence, by the definition of ϕ and ψ, we have that for $1 \leq j \leq L$,

$$u_\alpha \psi(u_j \otimes v'_j)(u_\alpha)^* \rightarrow \phi(u_j \otimes v'_j)$$

and

$$u_\alpha \psi(u_j \otimes v'_j)^*(u_\alpha)^* = u_\alpha \psi((u_j)^* \otimes (v'_j)^*)(u_\alpha)^* \rightarrow \phi((u_j)^* \otimes (v'_j)^*)$$

in the weak* topology on $\mathcal{M}$. Hence, since for all $\alpha \in I$ and for $1 \leq j \leq L$, $u_\alpha \psi(u_j \otimes v'_j)(u_\alpha)^*$, $\phi(u_j \otimes v'_j)$, $u_\alpha \psi(u_j \otimes v'_j)^*(u_\alpha)^*$, and $\phi((u_j)^* \otimes (v'_j)^*)$ are unitaries, we must have that for $1 \leq j \leq L$,

$$u_\alpha \psi(u_j \otimes v'_j)(u_\alpha)^* \to \phi(u_j \otimes v'_j)$$

in the σ-strong* topology in $\mathcal{M}$. We denote the above statement by "(V)".

Now since $\mathcal{E}$ is σ-strong* dense in $\mathcal{K}^{**} \cong \mathbb{B}(\mathcal{H})$, we have by (++++) and by [47] Theorem II.4.8, that $1_{\mathcal{A}} \otimes \mathcal{E} \otimes 1_{\mathcal{A}'}$ is σ-strong* dense in $1_{\mathcal{A}} \otimes \mathcal{K}^{**} \otimes 1_{\mathcal{A}'}$ (where we are referring to the σ-strong* topology on $\mathcal{M}$). Hence, for every α, $u_\alpha(1_{\mathcal{A}} \otimes \mathcal{E} \otimes 1_{\mathcal{A}'})(u_\alpha)^*$ is σ-strong* dense in $u_\alpha(1_{\mathcal{A}} \otimes \mathcal{K}^{**} \otimes 1_{\mathcal{A}'})(u_\alpha)^*$ (where again we are referring to the σ-strong* topology on $\mathcal{M}$). Hence, since $\mathcal{E}$ is a UHF algebra, since the norm topology on $\mathcal{M}$ is stronger than the σ-strong* topology on $\mathcal{M}$, and by [47] Theorem II.4.11, for every α, let $\{\mathcal{D}_\beta\}_{\beta \in I(\alpha)}$ be a net of simple finite dimensional unital C^*-subalgebras of $u_\alpha(1_{\mathcal{A}} \otimes \mathcal{E} \otimes 1_{\mathcal{A}'})(u_\alpha)^*$, and let $u_{j,\alpha,\beta}$ be a unitary in $\mathcal{D}_\beta$ for every $\beta \in I(\alpha)$ $(1 \leq j \leq L)$, such that

$$u_{j,\alpha,\beta} \to u_\alpha \psi(u_j \otimes v'_j)(u_\alpha)^*$$

$(1 \leq j \leq L)$ in the σ-strong* topology on $\mathcal{M}$. We denote the above statements by "(VV)".

Now let $\mathcal{P}$ be the collection of all finite subsets of the set of all normal states in $\mathcal{M}_*$. Consider the directed set $J =_{df} \{(\alpha, \beta, F) : \alpha \in I, \beta \in I(\alpha), F \in \mathcal{P}\}$, where the order structure is given by the following: $(\alpha, \beta, F) \leq (\alpha', \beta', F')$ if and only if all of the following hold:

- i. $\alpha \leq \alpha'$
- ii. Either $\alpha \neq \alpha'$, or $\alpha = \alpha'$ and $\beta \leq \beta'$
- iii. $F \subseteq F'$
- iv. For all $\rho \in F'$, for $1 \leq j \leq L$, $\|u_{j,\alpha',\beta'} - u_{\alpha'}\psi(u_j \otimes v'_j)(u_{\alpha'})^*\|^\sharp_{2,\rho} \leq \|u_{j,\alpha,\beta} - u_\alpha \psi(u_j \otimes v'_j)(u_\alpha)^*\|^\sharp_{2,\rho} + 1/(2|F'|) < 1/|F'|$, where $|F'|$ is the number of elements in F'.

By (V) and (VV), we have that J, with the above order structure, is a directed set. For $1 \leq j \leq L$, and for each $\gamma = (\alpha, \beta, F) \in J$, let $w_{j,\gamma}$ be the unitary in $\mathcal{M}$ that is given by $w_{j,\gamma} =_{df} u_{j,\alpha,\beta}$. By (V), (VV) and the definition of J, we have that for $1 \leq j \leq L$,

$$w_{j,\gamma} \to \phi(u_j \otimes v'_j)$$

in the strong* topology in $\mathcal{M}$. We denote the above statement by "(VVV)".

By the Choi-Effros Theorem ([8]), since O_2 is separable and nuclear, $(O_2)^{**}$ is a countably generated, injective von Neumann algebra. Hence, since $\mathcal{A}''_2 \cong O_2$, $(\mathcal{A}''_2)^{**}$ is a countably generated injective von Neumann algebra. Note that since $\mathcal{A}''_2 \cong O_2$ is properly infinite, $(\mathcal{A}''_2)^{**}$ is a properly infinite injective von Neumann algebra. Hence, by [26] and [21], and by [47] Theorem 4.11, let $\{\mathcal{E}_\kappa\}_{\kappa \in Q}$ be a net of finite dimensional unital $*$-subalgebras of $(\mathcal{A}''_2)^{**}$, and for $1 \leq j \leq L$, let $\{w'_{j,\kappa}\}_{\kappa \in Q}$ be a net of unitaries in $(\mathcal{A}''_2)^{**}$, such that $w'_{j,\kappa} \in \mathcal{E}_\kappa$ for all $\kappa \in Q$ and $w'_{j,\kappa} \to v''_j$ in the σ-strong* topology on $(\mathcal{A}''_2)^{**}$. From this and (++++), we must have that for $1 \leq j \leq L$,

$$1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}''_1} \otimes w'_{j,\kappa} \to 1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}''_1} \otimes v''_j$$

and

$$1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}''_1} \otimes (w'_{j,\kappa})^* \to 1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}''_1} \otimes (v''_j)^*$$

in the weak* topology on $\mathcal{M}$. But since all the quantities involved are unitaries, we must have that for $1 \leq j \leq L$,

$$1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes w'_{j,\kappa} \rightarrow 1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes v''_j$$

and

$$1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes (w'_{j,\kappa})^* \rightarrow 1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes (v''_j)^*$$

in the strong operator topology on $\mathcal{M}$. In other words,

$$1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes w'_{j,\kappa} \rightarrow 1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes v''_j$$

in the σ-strong* topology on $\mathcal{M}$. We denote the above statement by "(VVVV)".

Now consider the directed set $J \times Q$, where the order is given by $(\gamma, \kappa) \leq (\gamma', \kappa')$ if and only if $\gamma \leq \gamma'$ and $\kappa \leq \kappa'$, for all $(\gamma, \kappa), (\gamma', \kappa') \in J \times Q$. For $1 \leq j \leq L$, consider the net $\{w_{j,\gamma}(1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes w'_{j,\kappa})\}_{(\gamma,\kappa) \in J \times Q}$. By (VVV) and (VVVV), we must have that

$$w_{j,\gamma}(1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes w'_{j,\kappa}) \rightarrow \phi(u_j \otimes v'_j)(1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes v''_j)$$

in the σ-strong* topology on $\mathcal{M}$. Hence, by the definition of ϕ, we have that for $1 \leq j \leq L$,

$$w_{j,\gamma}(1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes w'_{j,\kappa}) \rightarrow u_j \otimes v'_j \otimes 1_{\mathcal{A}_1''} \otimes v''_j$$

in the σ-strong* topology on $\mathcal{M}$. We denote the above statement by "(VVVVV)".

By (VVVVV), let $(\gamma_0, \kappa_0) \in J \times Q$ such that for all $(\gamma, \kappa) \in J \times Q$ such that for $(\gamma, \kappa) \geq (\gamma_0, \kappa_0)$, and for $1 \leq i \leq k$ and $1 \leq j \leq L$, we have that

$$\|w_{j,\gamma}(1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes w'_{j,\kappa}) - u_j \otimes v'_j \otimes 1_{\mathcal{A}_1''} \otimes v''_j\|^{\sharp}_{2,\rho_i} < \delta/1000$$

From this and and (+++++++), we have that for $1 \leq i \leq k$ and $1 \leq j \leq L$,

$$\|w_{j,\gamma_0}(1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes w'_{j,\kappa_0}) - u_j \otimes u'_j \otimes u''_j\|^{\sharp}_{2,\rho_i} < 3\delta/1000$$

We denote the above statement by "(§)".

Now by the definition of w_{j,γ_0} (see (VVV), (VV) and (V)), there exist $\alpha_0 \in I$ and $\beta_0 \in I(\alpha_0)$, such that $w_{j,\gamma_0} \in \mathcal{D}_{\beta_0}$. Note that by the definition of $\mathcal{D}_{\beta_0}$ (see (VV)), and since $u_{\alpha_0} \in \mathcal{C} \subseteq \mathcal{A} \otimes \mathcal{K}^{**} \otimes \mathcal{A}_1'' \otimes 1_{\mathcal{A}_2''}$, $\mathcal{D}_{\beta_0}$ is a finite dimensional unital C^*-subalgebra of $\mathcal{M}$ such that

$$\mathcal{D}_{\beta_0} \subseteq u_{\alpha_0}(1_{\mathcal{A}} \otimes \mathcal{E} \otimes 1_{\mathcal{A}'})(u_{\alpha_0})^* \subseteq \mathcal{A} \otimes \mathcal{K}^{**} \otimes \mathcal{A}_1'' \otimes 1_{\mathcal{A}_2''}$$

Hence, $\mathcal{D}_{\beta_0}(1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes \mathcal{E}_{\kappa_0})$ must be a finite dimensional unital C^*-subalgebra of $\mathcal{M}$ such that for $1 \leq j \leq L$, $w_{j,\gamma_0}(1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes w'_{j,\kappa_0}) \in \mathcal{D}_{\beta_0}(1_{\mathcal{A}} \otimes 1_{\mathcal{K}^{**}} \otimes 1_{\mathcal{A}_1''} \otimes \mathcal{E}_{\kappa_0})$. From this and (§), we have proven statement (+++).

From statement (+++) and by Lemma 4.3, $\mathcal{A}^{**} \overline{\otimes} \mathbb{B}(\mathcal{H}) \overline{\otimes} (\mathcal{A}')^{**}$ is an injective von Neumann algebra. In other words, we have proven Claim 1.

By [47] Theorem V.1.19, $(\mathcal{A}')^{**}$ has a finite direct sum decomposition

$$(\mathcal{A}')^{**} \cong \mathcal{M}_I \oplus \mathcal{M}_{II} \oplus \mathcal{M}_{III}$$

where $\mathcal{M}_I$ is a type I von Neumann algebra, $\mathcal{M}_{II}$ is a type II von Neumann algebra, and $\mathcal{M}_{III}$ is a type III von Neumann algebra. Hence,

$$\mathcal{A}^{**} \overline{\otimes} \mathbb{B}(\mathcal{H}) \overline{\otimes} (\mathcal{A}')^{**} \cong (\mathcal{A}^{**} \overline{\otimes} \mathbb{B}(\mathcal{H}) \overline{\otimes} \mathcal{M}_I) \oplus (\mathcal{A}^{**} \overline{\otimes} \mathbb{B}(\mathcal{H}) \overline{\otimes} \mathcal{M}_{II}) \oplus (\mathcal{A}^{**} \overline{\otimes} \mathbb{B}(\mathcal{H}) \overline{\otimes} \mathcal{M}_{III})$$

Since injectivity is preserved under completely contractive projections, we have that $\mathcal{A}^{**} \overline{\otimes} \mathbb{B}(\mathcal{H}) \overline{\otimes} \mathcal{M}_I$ is injective. But by [47] Theorem V.1.27, and since (once more) injectivity is preserved under completely contractive projections, there exist an infinite dimensional Hilbert space $\mathcal{H}'$ and a commutative von Neumann algebra $\mathcal{B}$ such that $\mathcal{A}^{**}\overline{\otimes}\mathbb{B}(\mathcal{H})\overline{\otimes}\mathcal{B}\overline{\otimes}\mathbb{B}(\mathcal{H}')$ is an injective von Neumann algebra. By cutting this von Neumann algebra down by a projection of the form $1_{\mathcal{A}^{**}} \otimes p \otimes 1_{\mathcal{B}} \otimes q$, where $p \in \mathbb{B}(\mathcal{H})$, $q \in \mathbb{B}(\mathcal{H}')$ are minimal projections, and since (again), injectivity is preserved under completely contractive projections, we see that $\mathcal{A}^{**}\overline{\otimes}\mathcal{B}$ is an injective von Neumann algebra. Since $\mathcal{B}$ is a commutative von Neummann algebra, and by [33] Theorem 11.2.9, there is a completely contractive projection of $\mathcal{A}^{**}\overline{\otimes}\mathcal{B}$ onto $\mathcal{A}^{**} \otimes 1_{\mathcal{B}}$ (just use [33] Theorem 11.2.9 and any unital $*$-homomorphism from $\mathcal{B}$ onto the complex numbers $\mathbb{C}$). Hence, since injectivity is preserved under completely contractive projections, $\mathcal{A}^{**}$ is injective. Hence, by the Choi-Effros Theorem (see [8]), $\mathcal{A}$ is a nuclear C^*-algebra. □

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