# The Range of a Class of Classifiable Separable Simple Amenable $C^*$-algebras

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**Abstract**

We study the range of a classifiable class $\mathcal{A}$ of unital separable simple amenable $C^*$-algebras which satisfy the Universal Coefficient Theorem. The class $\mathcal{A}$ contains all unital simple AH-algebras. We show that all unital simple inductive limits of dimension drop circle $C^*$-algebras are also in the class. This unifies some of the previous known classification results for unital simple amenable $C^*$-algebras. We also show that there are many other $C^*$-algebras in the class. We prove that, for any partially ordered, simple weakly unperforated rationally Riesz group $G_0$ with order unit $u$, any countable abelian group $G_1$, any metrizable Choquet simplex $S$, and any surjective affine continuous map $r : S \to S_u(G_0)$ (where $S_u(G_0)$ is the state space of $G_0$) which preserves extremal points, there exists one and only one (up to isomorphism) unital separable simple amenable $C^*$-algebra $A$ in the classifiable class $\mathcal{A}$ such that

$$((K_0(A), K_0(A)_+, [1_A]), K_1(A), T(A), \lambda_A) = ((G_0, (G_0)_+, u), G_1, S, r).$$

## 1 Introduction

Recent years saw some rapid development in the theory of classification of amenable $C^*$-algebras, or otherwise know as the Elliott program of classification of amenable $C^*$-algebras. One of the high lights is the Kirchberg-Phillips's classification of separable purely infinite simple amenable $C^*$-algebras which satisfy the Universal Coefficient Theorem (see [21] and [12]). There are also exciting results in the simple $C^*$-algebras of stable rank one. For example, the classification of unital simple AH-algebra with no dimension growth by Elliott, Gong and Li ([7]). Limitation of the classification have been also discovered (see [24] and [26], for example). In particular, it is now known that the general class of unital simple AH-algebras can not be classified by the traditional Elliott invariant. One crucial condition must be assumed for any general classification (using the Elliott invariant) of separable simple amenable $C^*$-algebras is the $\mathcal{Z}$-stability. On the other hand, classification theorems were established for unital separable simple amenable $C^*$-algebras which are not assumed to be AH-algebras, or other inductive limit structures (see [14], [17] and [20]). Winter's recent result provided a new approach to some more general classification theorem ([30] and [16]). Let $\mathcal{A}$ be the class of unital separable simple amenable $C^*$-algebras $A$ which satisfy the UCT for which $A \otimes M_{\mathfrak{p}}$ has tracial rank no more than one for some supernatural number $\mathfrak{p}$ of infinite type. A more recent work in [18] shows that $C^*$-algebras in $\mathcal{A}$ can be classified by the Elliott invariant up to $\mathcal{Z}$-stable isomorphism. All unital simple AH-algebras are in $\mathcal{A}$. One consequence of this is now we know that classifiable class of unital simple AH-algebras is exact the class of those of $\mathcal{Z}$-stable ones. But class $\mathcal{A}$ contains more unital simple $C^*$-algebras. Any unital separable simple ASH-algebra $A$ whose state space $S(K_0(A))$ of its $K_0(A)$ is the same as that tracial state space are in $\mathcal{A}$. It also contains the Jiang-Su algebra $\mathcal{Z}$ and many other projectionless simple $C^*$-algebras. We show that the class contains all unital simple so-called dimension drop circle algebras as well as many other $C^*$-algebras whose $K_0$-groups are not Riesz. It is the purpose of this paper to discuss the range of invariant of $C^*$-algebras in $\mathcal{A}$.

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## 2 Preliminaries

**Definition 2.1** (Dimension drop interval algebras [11])**.** A dimension drop interval algebra is a $C^*$-algebra of the form:

$$\mathbf{I}(m_0, m, m_1) = \{f \in \mathrm{C}([0,1], M_m) : f(0) \in M_{m_0} \otimes 1_{m/m_0} \text{ and } f(1) \in 1_{m/m_1} \otimes M_{m_1}\},$$

where $m_0, m_1$ and $m$ are positive integers with $m$ is divisible by $m_0$ and $m_1$. If $m_0$ and $m_1$ are relatively prime, and $m = m_0 m_1$, then $\mathbf{I}(m_0, m, m_1)$ is called a prime dimension drop algebra.

**Definition 2.2** (The Jiang-Su algebra [11])**.** Denote by $\mathcal{Z}$ the Jiang-Su algebra of unital infinite dimensional simple $C^*$-algebra which is an inductive limit of prime dimension drop algebras with a unique tracial state, $(K_0(\mathcal{Z}), K_0(\mathcal{Z})_+, [1_{\mathcal{Z}}]) = (\mathbb{Z}, \mathbb{N}, 1)$ and $K_1(\mathcal{Z}) = 0$.

**Definition 2.3** (Dimension drop circle algebras [19])**.** Let $n$ be a natural number. Let $x_1, ..., x_N$ be points in the circle $\mathbb{T}$, and let $d_1, ..., d_N$ be natural numbers dividing $n$. Then a dimension drop circle algebra is a $C^*$-algebra of the form

$$D(n, d_1, ..., d_N) = \{f \in \mathrm{C}(\mathbb{T}, \mathrm{M}_N) : \ f(x_i) \in M_{d_i} \otimes 1_{n/d_i}, i = 1, 2, ..., N\}.$$

**Definition 2.4** (Simple A$\mathbb{T}$D-algebras)**.** By A$\mathbb{T}$D-algebras we mean $C^*$-algebras which are inductive limits of dimension drop circle algebras.

**Definition 2.5.** Denote by $\mathcal{I}$ the class of those $C^*$-algebras with the form $\bigoplus_{i=1}^n M_{r_i}(C(X_i))$, where each $X_i$ is a finite CW complex with (covering) dimension no more than one.

A unital simple $C^*$-algebra $A$ is said to have tracial rank one if for any finite subset $\mathcal{F} \subset A$, $\epsilon > 0$, any nonzero positive element $a \in A$, there is a $C^*$-subalgebra $C \in \mathcal{I}$ such that if denote by $p$ the unit of $C$, then for any $x \in \mathcal{F}$, one has

(1) $||xp - px|| \leq \epsilon$,

(2) there is $b \in C$ such that $||b - pxp|| \leq \epsilon$ and

(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

Denote by $\mathcal{I}'$ the class of all unital $C^*$-algebras with the form $\bigoplus_{i=1}^n M_{r_i}(C(X_i))$, where each $X_i$ is a compact metric space with dimension no more than one.

Note that, in the above definition, one may replace $\mathcal{I}$ by $\mathcal{I}'$.

**Definition 2.6** (A classifiable class of unital separable simple amenable $C^*$-algebras)**.** Denote by $\mathcal{N}$ the class of all unital separable amenable $C^*$-algebras which satisfy the Universal Coefficient Theorem.

For a supernatural number $\mathfrak{p}$, denote by $M_{\mathfrak{p}}$ the UHF algebra associated with $\mathfrak{p}$ (see [2]).

Let $\mathcal{A}$ denote the class of all unital separable simple amenable $C^*$-algebras $A$ in $\mathcal{N}$ for which $TR(A \otimes M_{\mathfrak{p}}) \leq 1$ for all supernatural numbers $\mathfrak{p}$ of infinite type.

**Remark 2.7.** By Theorem 2.11 below, in order to verify whether a C*-algebra $A$ is in the class $\mathcal{A}$, it is enough to verify $TR(A \otimes M_{\mathfrak{p}}) \leq 1$ for one supernatural number $\mathfrak{p}$ of infinite type.

**Definition 2.8.** Let $G$ be a partially ordered group with an order unit $u \in G$. Denote by $S_u(G)$ the state space of $G$, i.e., $S_u(G)$ is the set of all positive homomorphisms $h : G \to \mathbb{R}$ such that $h(u) = 1$. The set $S_u(G)$ equipped with the weak-*-topology forms a compact convex set. Denote by $\mathrm{Aff}(S_u(G))$ the space of all continuous real affine functions on $S_u(G)$. We use $\rho$ for the homomorphism $\rho : G \to \mathrm{Aff}(S_u(G))$ defined by

$$\rho(g)(s) = s(g) \text{ for all } s \in S_u(G) \text{ and } \text{ for all } g \in G.$$

Put $\mathrm{Inf}(G) = \ker\rho$.

**Definition 2.9.** Let $A$ be a unital stably finite separable simple amenable $C^*$-algebra. Denote by $T(A)$ the tracial state space of $A$. We also use $\tau$ for $\tau \otimes \mathrm{Tr}$ on $A \otimes M_k$ for any integer $k \geq 1$, where $Tr$ is the standard trace on $M_k$.

By $\mathrm{Ell}(A)$ we mean the following:

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A),$$

where $r_A : T(A) \to S_{[1_A]}(K_0(A))$ is a surjective continuous affine map such that $r_A(\tau)([p]) = \tau(p)$ for all projections $p \in A \otimes M_k$, $k = 1, 2, \ldots$.

Suppose that $B$ is another stably finite unital separable simple $C^*$-algebra. A map $\Lambda$ : $\mathrm{Ell}(A) \to \mathrm{Ell}(B)$ is said to be a homomorphism if $\Lambda$ gives an order homomorphism $\lambda_0 : K_0(A) \to K_0(B)$ such that $\lambda_0([1_A]) = [1_B]$, a homomorphism $\lambda_1 : K_1(A) \to K_1(B)$, a continuous affine map $\lambda'_\rho : T(B) \to T(A)$ such that

$$\lambda'_\rho(\tau)(p) = r_B(\tau)(\lambda_0([p]))$$

for all projection in $A \otimes M_k$, $k = 1, 2, \ldots$, and for all $\tau \in T(B)$.

We say that such $\Lambda$ is an isomorphism, if $\lambda_0$ and $\lambda_1$ are isomorphisms and $\lambda'_\rho$ is a affine homeomorphism. In this case, there is an affine homeomorphism $\lambda_\rho : T(A) \to T(B)$ such that $\lambda_\rho^{-1} = \lambda'_\rho$.

**Theorem 2.10** (Corollary 11.9 of [18])**.** *Let $A$, $B \in \mathcal{A}$. Then*

$$A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$$

*if*

$$\mathrm{Ell}(A \otimes \mathcal{Z}) = \mathrm{Ell}(B \otimes \mathcal{Z}).$$

In the next section (Theorem 3.6), we will show the following:

**Theorem 2.11.** *Let $A$ be a unital separable amenable simple $C^*$-algebra. Then $A \in \mathcal{A}$ if only if $TR(A \otimes M_{\mathfrak{p}}) \leq 1$ for one supernatural number $\mathfrak{p}$ of infinite type*

**Definition 2.12.** Recall that a $C^*$-algebra $A$ is said to be $\mathcal{Z}$-stable if $A \otimes \mathcal{Z} \cong A$. Denote by $\mathcal{A}_{\mathcal{Z}}$ the class of $\mathcal{Z}$-stable $C^*$-algebras in $\mathcal{A}$. Denote by $\mathcal{A}_{0z}$ the subclass of those $C^*$-algebras $A \in \mathcal{A}_z$ for which $TR(A \otimes M_{\mathfrak{p}}) = 0$ for some supernatural number $\mathfrak{p}$ of infinite type.

**Corollary 2.13.** *Let $A$ and $B$ be two unital separable amenable simple $C^*$-algebras in $\mathcal{A}_{\mathcal{Z}}$. Then $A \cong B$ if and only if*

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

**Definition 2.14.** Let $A$ be a unital simple $C^*$-algebra and let $a, b \in A_+$. We define

$$a \lesssim b,$$

if there exists $x \in A$ such that $x^*x = a$ and $xx^* \in \overline{bAb}$. We write $[a] = [b]$ if there exists $x \in A$ such that $x^*x = a$ and $xx^* = b$. We write $[a] \le [b]$ if $a \lesssim b$.

If $e \in A$ is a projection and $[e] \le [a]$, then there is a projection $p \in \overline{aAa}$ such that $e$ is equivalent to $p$.

If there $n$ mutually orthogonal elements $b_1, b_2, ..., b_n \in \overline{bAb}$ such that $a \lesssim b_i$, $i = 1, 2, ..., n$, then we write

$$n[a] \le [b].$$

Let $m$ be another positive integer. We write

$$n[a] \le m[b],$$

if, in $M_m(A)$,

$$n[a] \le [\begin{pmatrix} b & 0 & 0 \cdots & 0 \\ 0 & b & 0 \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 \cdots & b \end{pmatrix}],$$

where $b$ repeats $m$ times.

## 3 The classifiable $C^*$-algebras $\mathcal{A}$

The purpose of this section is to provide a proof of Theorem 2.11.

**Lemma 3.1.** *Let $A$ be a unital simple $C^*$-algebra with an increasing sequence of unital simple $C^*$-algebras $\{A_n\}$ such that $1_A = 1_{A_n}$ and $\cup_{n=1}^{\infty} A_n$ is dense in $A$.*

*Suppose that $a \in A_+ \setminus \{0\}$. Then, there exists $b \in (A_n)_+ \setminus \{0\}$ for some large $n$ so that $b \lesssim a$.*

*Proof.* Without loss of generality, we may assume that $\|a\| = 1$. Let $1 > \delta > 0$. There exists $\epsilon > 0$ such that for any $c \in A_+ \setminus \{0\}$ with

$$\|a - c\| < \epsilon$$

one has (see Proposition 2.2 of [23]) that

$$f_\delta(c) \lesssim a,$$

where $f_\delta \in C_0((0, \infty))$ for which $f_\delta(t) = 1$ for all $t \ge \delta$ and $f_\delta(t) = 0$ for all $t \in (0, \delta/2)$. We may assume, for a sufficiently small $\epsilon$, $f_\delta(c) \ne 0$. Since $\cup_{n=1}^{\infty} A_n$ is dense in $A$, it is possible to such that $c \in A_n$ for some large $n$. Put $b = f_\delta(c)$. Then $b \in A_n$ and $b \lesssim a$.

□

**Lemma 3.2.** *Let $A$ be a unital simple $C^*$-algebra and $e, a \in A_+ \setminus \{0\}$. Then, for any integer $n > 0$, there exists $m(n) > 0$ such that*

$$n[e] \le m(n)[a].$$

*Proof.* It suffices to show that $[1_A] \le m[a]$ for some integer $m$. Since $A$ is simple, there are $y_1, y_2, ..., y_m \in A$ for some integer $m$ such that

$$\sum_{i=1}^{m} y_i^* a y_i = 1$$

(see, for example, Lemma 3.3.6 of [13]). Let $b_i' = a^{1/2} y_i y_i^* a^{1/2}$, $i = 1, 2, ..., m$ Define

$$b_i = \mathrm{diag}(\overbrace{0, 0, ..., 0}^{i-1}, b_i, 0, ..., 0), \quad i = 1, 2, ..., m$$

in $M_m(A)$. Define

$$y = \begin{pmatrix} y_1^* a^{1/2} & y_2^* a^{1/2} & \cdots & y_m^* a^{1/2} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then

$$yy^* = 1_A \text{ and } y^* y = \sum_{i=1}^{m} b_i.$$

Thus

$$[1_A] \le [\sum_{i=1}^{m} b_i].$$

Clearly

$$[\sum_{i=1}^{m} b_i] \le m[a].$$

□

**Lemma 3.3.** *Let $A$ be a unital infinite dimensional simple $C^*$-algebra and let $a \in A_+ \setminus \{0\}$. Suppose that $n \ge 1$ is an integer. Then there are non-zero mutually orthogonal elements $a_1, a_2, ..., a_n \in \overline{aAa}_+$ such that*

$$a_1 \lesssim a_2 \lesssim \cdots \lesssim a_n.$$

*Proof.* In $\overline{aAa}$, there exists a positive element $0 \le b \le 1$ such that its spectrum has infinitely many points. From this one obtains $n$ non-zero mutually orthogonal positive elements $b_1, b_2, ..., b_n$. Since $A$ is simple, there is $x_{n-1} \in A$ such that $b_{n-1} x_{n-1} b_n \ne 0$. Set $y_{n-1} = b_{n-1} x_{n-1} b_n$. One then has that $y_{n-1}^* y_{n-1} \in \overline{b_n A b_n}$ and $y_{n-1} y_{n-1}^* \in \overline{b_{n-1} A b_{n-1}}$. Then consider $b_1, b_2, ..., b_{n-2}, y_{n-1} y_{n-1}^*$. The lemma follows by applying the argument above $n-1$ more times. □

**Lemma 3.4.** *Let $A$ be a unital separable simple $C^*$-algebra. Suppose that $\{A_n\}$ be an increasing sequence of unital simple $C^*$-algebras such that $1_A = 1_{A_n}$ and $\cup_{n=1}^{\infty} A_n$ is dense in $A$.*

*Then $TR(A) \le 1$ if and only if the following holds: For any $\epsilon > 0$, any integer $n \ge 1$, any $a \in (A_n)_+ \setminus \{0\}$ and a finite subset $\mathcal{F} \subset A_n$, there exists an integer $N \ge 1$ satisfying the following: There is, for each $m \ge N$, a $C^*$-subalgebra $C \subset A_m$ with $C \in \mathcal{I}'$ and with $1_C = p$ such that*

(1) $\|px - xp\| < \epsilon$ *for all* $x \in \mathcal{F}$;
(2) $\mathrm{dist}(pxp, C) < \epsilon$ *for all* $x \in \mathcal{F}$ *and*
(3) $1 - p$ *is equivalent to a projection* $q \in \overline{aA_m a}$.

*Proof.* First we note that "if " part follows the definition easily.

To prove the "only if " part, we assume that $TR(A) \le 1$ and there is an increasing sequence of $C^*$-subalgebras $A_n \subset A$ which are unital and the closure of the union $\cup_{n=1}^{\infty} A_n$ is dense in $A$. In particular, we may assume that $1_{A_n} = 1_A$, $n = 1, 2, ....$

Let $\epsilon > 0$, $n \ge 1$, $a \in (A_n)_+ \setminus \{0\}$ and a finite subset $\mathcal{F} \subset A_n$. Since $TR(A) \le 1$, there is a $C^*$-subalgebra $C_1 \in \mathcal{I}$ with $1_{C_1} = q$ such that

(1) $\|xq - qx\| < \epsilon/2$ for all $x \in \mathcal{F}$;

(2) $\mathrm{dist}(qxq, C_1) < \epsilon/2$ for all $x \in \mathcal{F}$ and

(3) $[1 - q] \le [a]$.

From (2) above, there is a finite subset $\mathcal{F}_1 \subset C_1$ such that

(2') $\mathrm{dist}(qxq, \mathcal{F}_1) < \epsilon/2$ for all $x \in \mathcal{F}$.

We may assume that $q \in \mathcal{F}_1$.

Put $d = \sup\{\|x\|; x \in \mathcal{F}\}$.

There is $\delta > 0$ and a finite subset $\mathcal{G} \subset C_1$ satisfying the following: If $B \subset A$ is a unital $C^*$-subalgebra and

$$\mathrm{dist}(y, B) < \delta$$

for all $y \in \mathcal{G}$, there is a $C^*$-subalgebra $C' \in \mathcal{I}'$ such that $C' \subset B$ and

$$\mathrm{dist}(z, C') < \epsilon/2(d + 1)$$

for all $z \in \mathcal{F}_1 \cup \mathcal{G}$.

By the assumption, there is an integer $N \ge n$ such that

$$\mathrm{dist}(y, A_N) < \delta/2$$

for all $y \in \mathcal{G}$. It follows that there is $C \in \mathcal{I}'$ with $1_C = p$ such that $C \in A_N$ and

$$\mathrm{dist}(z, C) < \epsilon/2(d + 1)$$

for all $z \in \mathcal{F}_1 \cup \mathcal{G}$. In particular,

$$\|p - q\| < \epsilon/2(d + 1).$$

One then checks that

(i) $\|qx - xq\| < \epsilon$ for all $x \in \mathcal{F}$;

(ii) $\mathrm{dist}(pxp, C) < \epsilon$ for all $x \in \mathcal{F}$ and

(iii) $[1 - p] = [1 - q] \le [a]$.

The lemma follows.

□

The following is well-known.

**Lemma 3.5.** *For any two positive integers $p$ and $q$ with $p > q$. Then there exist integers $1 \le m < q$, $0 \le r < p$ and $s \ge 1$ such that*

$$p = mq^s + r.$$

*Proof.* There exists a largest integer $s \ge 1$ such that $p \ge q^s$. There are integers $0 \le r < p$ and $1 \le m$ such that

$$p = mq^s + r.$$

Then $1 \le m < q$.

□

Theorem 2.11 follows immediately from the following:

**Theorem 3.6.** *Let $A$ be a unital separable simple $C^*$-algebra. Then $TR(A\otimes M_{\mathfrak{p}})\le 1$ for all supernatural numbers $\mathfrak{p}$ of infinite type if and only if there exists one supernatural number $\mathfrak{q}$ of infinite type such that $TR(A\otimes M_{\mathfrak{q}})\le 1$.*

*Proof.* Suppose that there is a supernatural number $\mathfrak{q}$ of infinite type such that $TR(A\otimes M_{\mathfrak{q}})\le 1$.

Let $A_n\cong A\otimes M_{r(n)}$ such that $1_{A_n}=1_{A\otimes M_{\mathfrak{p}}}$ and $\cup_{n=1}^{\infty}A_n$ is dense in $A\otimes M_{\mathfrak{p}}$. Let $B_n\cong A\otimes M_{k(n)}$ such that $1_{B_n}=1_{A\otimes M_{\mathfrak{q}}}$ and $\cup_{n=1}^{\infty}B_n$ is dense in $A\otimes M_{\mathfrak{p}}$.

Write $\mathfrak{q}=q_1^{\infty}q_2^{\infty}\cdots q_k^{\infty}\cdots$, where $q_1,q_2,...,q_k,...$ are prime numbers. We may assume that $1<q_1<q_2<\cdots q_k<\cdots$.

Fix $\epsilon>0$, $n\ge 1$, $a_0\in (A_n)_+\setminus\{0\}$ and $\mathcal{F}\subset A_n$. Since $A_n$ is simple, there are mutually orthogonal elements $a_1,a_2,...,a_{3(q_1+1)}\in (A_n)_+\setminus\{0\}$ such that $a_1,a_2,...,a_{3(q_1+1)}\in\overline{a_0A_na_0}$ and $a_1\lesssim a_2\lesssim\cdots\lesssim a_{3(q_1+1)}$.

By 3.1, there exists an integer $m(1)\ge 1$ such that

$$[1_{A_n}]\le m(1)[a_{q_1+1}].$$

Write $A_n\cong M_{r(n)}(A)$.

Note that $TR(M_{r(n)}(A)\otimes M_{\mathfrak{q}})\le 1$. Therefore, by 3.4, there exists $N\ge 1$ satisfying the following: there is, for each $m\ge N$, a $C^*$-subalgebra $C_m\subset M_{r(n)}(A)\otimes M_{k(m)}$ with $C_m\in\mathcal{I}'$ and $1_C=e(m)$ such that

(i) $\|e(m)j_0(x)-j_0(x)e(m)\|<\epsilon/2$ for all $x\in\mathcal{F}$, where $j_m: M_{r(n)}(A)\to M_{r(n)}(A)\otimes M_{k(m)}$ is defined by $j_m(a)=a\otimes 1_{M_{k(m)}}$ for $a\in M_{r(n)}(A)$;

(ii) $\mathrm{dist}(e(m)j_m(x)e(m),C)<\epsilon/2$ for all $x\in\mathcal{F}$ and

(iii) $1_{M_{r(n)}(A)\otimes M_{r(m)}}-e(m)$ is equivalent to a projection in $\overline{j_m(a_1)(A_n\otimes M_{k(m)})j_m(a_1)}$.

We may assume that $k(N)=q_1^{s_1}q_2^{s_2}\cdots q_k^{s_k}$ and $k(N)\ge m(q_1)$.

To simplify the notation, without loss of generality, we may assume that $r(n+1)/r(n)>k(N)$. We write

$$\frac{r(n+1)}{r(n)}=N_1k(N)+r_0,$$

where $N_1\ge 1$and $0\le r_0<k(N)$ are integers. Without loss of generality, we may further assume that $N_1>m(1)k(N)$. We write

$$N_1=n_1q_1^s+r_1,$$

where $q_1>n_1\ge 1$, $s\ge 1$, and $0\le r_1<q_1$ are integers. Thus

$$\frac{r(n+1)}{r(n)}=n_1q_1^sK(N)+r_1K(N)+r_0.$$

Without loss of generality, to simplify notation, we may assume that $k(N+1)=q_1^sK(N)$.

Put $C=\bigoplus_{j=1}^{n_1}C_{N+1}\oplus\bigoplus_{i=1}^{r_1}C_N$. Put $e=\bigoplus_{j=1}^{n_1}e(N+1)\oplus\bigoplus_{i=1}^{r_1}e(N)$. Define $j': M_{r(n)}(A)\to M_{r(n)}(A)\otimes M_{r(n+1)/r(n)-r_0}$ by

$$j'(a)=\mathrm{diag}(\overbrace{j_{N+1}(a),j_{N+1}(a),\cdots,j_{N+1}(a)}^{n_1})\oplus\mathrm{diag}(\overbrace{j_N(a),j_N(a),\cdots,j_N(a)}^{r_1})$$

for all $a\in M_{r(n)}(A)$.

By what we have proved, we have that

$\|ej'(x)-j'(x)e\|<\epsilon/2$ for all $x\in\mathcal{F}$,

$\mathrm{dist}(ej'(x)e,C)<\epsilon/2$ for all $x\in\mathcal{F}$ and

$1_{M_{r(n)}(A)\otimes M_{r(n+1)/r(n)}-r_0} - e$ is equivalent to a projection in $\overline{b_1(M_{r(n)}(A)\otimes M_{r(n+1)/r(n)-r_0})b_1}$, where

$$b_1 = \sum_{i=1}^{n_1} j'(a_i) + \sum_{i=p_1+1}^{p_1+r_1} j'(a_i).$$

The last assertion follows the fact that

$$n_1[1_{M_{r(n)}(A)\otimes M_{k(N+1)}} - e(N+1)] \le [\sum_{i=1}^{n_1} a_i] \text{ and}$$

$$r_1[1_{M_{r(n)}(A)\otimes M_{k(N)}} - e(N)] \le [\sum_{i=p_1+1}^{p_1+r_1} a_i].$$

Define $j : M_{r(n)}(A) \to M_{r(n+1)}(A) = M_{r(n)}(A) \otimes M_{r(n+1)/r(n)}$ by

$$j(a) = a \otimes 1_{M_{r(n+1)/r(n)}} \text{ for all } a \in M_{r(n)}.$$

Thus, in $M_{r(n+1)}(A)$,
$\|ej(a) - j(a)e\| < \epsilon/2$ for all $x \in \mathcal{F}$,
$\mathrm{dist}(ej(a)e, C) < \epsilon/2$ for all $x \in \mathcal{F}$.
Note that

$$r_0[1_{M_{r(n)}}] \le r_0 m(1)[a_1] \le [j(a_{2q_1+1})].$$

Thus

$$[1_{A_{n+1}} - e] = r_0[1_{M_{r(n)}}] + [1_{M_{r(n)}(A)\otimes M_{r(n+1)/r(n)}-r_0} - e] \le [\sum_{i=1}^{3(q_1+1)} a_i].$$

It follows that $1_{A_{n+1}} - e$ is equivalent to a projection in $\overline{j(a_0)A_{n+1}j(a_0)}$.

This proves that $TR(M_{r(n)}(A) \otimes M_{\mathfrak{p}}) \le 1$. It follows from Proposition 3.2 of [17] that $TR(A \otimes M_{\mathfrak{p}}) \le 1$.

□

**Corollary 3.7.** *Let $A$ be a unital separable simple $C^*$-algebra. Suppose that there exists a supernatural number $\mathfrak{q}$ of infinite type for which $TR(A \otimes M_{\mathfrak{q}}) = 0$. Then, for all supernatural number $\mathfrak{p}$ of infinite type, $TR(A \otimes M_{\mathfrak{p}}) = 0$.*

*Proof.* The proof uses the exactly the same argument used in the proof of 3.6 (and 3.4). □

# 4 Unital Simple A$\mathbb{T}$D-algebras

**Theorem 4.1.** *Every unital simple A$\mathbb{T}$D-algebra $A$ for which $K_0(A)/\mathrm{ker}\rho_A \ncong \mathbb{Z}$ has tracial rank one or zero.*

*Proof.* Since $K_1(A)$ is a countable abelian group, we can write

$$K_1(A) = \varinjlim_{n\to\infty} (G_n, \iota_n),$$

with $G_n \cong \bigoplus_{l=1}^{l_n} \mathbb{Z}/p_{n,l}\mathbb{Z}$ for some non-negative integers $l_n$ and $p_{n,l}$, where $p_{n,l} \neq 1$ (note that if $p_{n,l} = 0$, one has that $\mathbb{Z}/p_{n,l}\mathbb{Z} \cong \mathbb{Z}$). Denote by $p_n = \sum_{l=1}^{l_n} p_{n,l}$.

Since $K_0(A)$ is a simple Riesz group and $K_0(A)/\mathrm{ker}\rho_A \ncong \mathbb{Z}$, and the pairing between $T(A)$ and $K_0(A)$ preserves extreme points, by [28], there is a simple inductive limit of interval algebras $B$ such that

$$((K_0(A), K_0(A)_+, [1_A]), T(A), \lambda_A) \cong ((K_0(B), K_0(B)_+, [1_B]), T(B), \lambda_B). \tag{e 4.1}$$

Write

$$B = \varinjlim_{n\to\infty} (\bigoplus_{i=1}^{k_n} B_{n,i}, \varphi_n),$$

where $B_{n,i} = \mathrm{M}_{m_{n,i}}(\mathrm{C}([0,1]))$, and write $[\varphi_n]_0 = (r_{n,i,j})$ with $r_{n,i,j} \in \mathbb{Z}^+$, where $1 \leq i \leq k_{n+1}$ and $1 \leq j \leq k_n$. Since $A$ is simple, without loss of generality, we may assume that $r_{n,1,j} > (n+1)p_n$ by passing to a subsequence.

For each $n$ and each $1 \leq j \leq k_{n+1}$, consider the restriction of the map $\varphi_n$ to the direct summand $B_{n,1}$ and $B_{n+1,j}$, that is, consider the map $\varphi_n(1,j) : B_{n,1} \to B_{n+1,j}$. It follows from [4] that we may assume that there exist continuous functions $s_1, s_2, ..., s_{r_{n,1,j}}$ such that

$$\varphi_n(1,j)(f)(t) = W^*(t)\mathrm{diag}\{f \circ s_1(t), ..., f \circ s_{r_{n,1,j}}(t)\}W(t)$$

for some $W(t) \in \mathrm{M}_{m_{n,i}r_{n,1,j}}(\mathrm{C}([0,1]))$. Factor through the map $\varphi_n(1,j)$ by

$$B_{n,1} \xrightarrow{\psi_1} (\textstyle\bigoplus_{l=1}^{l_n} \mathrm{M}_{p_{n,l}m_{n,1}}(\mathrm{C}([0,1]))) \oplus B_{n,1} \xrightarrow{\psi_2} B_{n+1,j},$$

where

$$\psi_1(f)(t) = \mathrm{diag}\{\{f \circ s_1, ..., f \circ s_{p_{n,1}}\}, ..., \{f \circ s_{p_n - p_{n,l_n}+1}, ..., f \circ s_{p_n}\}, f\}.$$

and

$$\psi_2(f \oplus g) = W^*\mathrm{diag}\{f, g \circ s_{p_n+1}, ..., g \circ s_{r_{n,1,j}}\}W$$

is the diagonal embedding. Since $r_{n,1,j} \geq np_n$, one has that the restriction of any tracial state of $B$ to the unit of $\bigoplus_{l=1}^{l_n} \mathrm{M}_{p_{n,l}m_{n,1}}(\mathrm{C}([0,1]))$ has value less than $1/n$.

Therefore, by replacing $B_{n,1}$ by $\mathrm{M}_{p_{n,l}m_{n,1}}(\mathrm{C}([0,1])) \oplus (\bigoplus_{r_{n,1,j}-p_n} B_{n,1})$, one may assume that there is an inductive limit decomposition

$$B = \varinjlim_{n\to\infty} (\bigoplus_{i=1}^{k_n} B_{n,i}, \varphi_n),$$

where $B_{n,i} = \mathrm{M}_{m_{n,i}}(\mathrm{C}([0,1]))$, such that $k_n > l_n$, and $m_{n,j} = d_{n,j}p_{n,j}$ for some natural number $d_{n,j}$ for any $1 \leq j \leq l_n$. Moreover, one has that $\tau(e) < 1/n$ for any $\tau \in T(A)$, where $e$ is the unit of $\bigoplus_{j=1}^{l_n} B_{n,j}$. In other words,

$$\rho(e) < 1/n \tag{e 4.2}$$

for any $\rho \in S_u(K_0(A))$. Fix this inductive limit decomposition.

Now, let us replace certain interval algebras at each level $n$ by certain dimension drop interval algebras so that the new inductive limit gives the desired $K_1$-group, and keep the $K_0$-group and the paring unchanged.

At level $n$, for each $1 \leq l \leq l_n$, if $p_{n,l} \neq 0$, denote by $D_{n,l}$ the dimension drop C*-algebra $\mathbf{I}[m_{n,l}, m_{n,l}p_{n,l}, m_{n,l}]$; if $p_{n,l} = 0$, denote by $D_{n,l}$ the circle algebra $\mathrm{M}_{m_{n,l}}(\mathrm{C}(\mathbb{T}))$. Then, one has

$$(K_0(D_{n,l}), K^+(D_{n,l}), 1_{D_{n,l}}) = (\mathbb{Z}, \mathbb{Z}^+, m_{n,l}) \quad \text{and} \quad K_1(D_{n,l}) = \mathbb{Z}/p_{n,l}\mathbb{Z}.$$

Replace each $B_{n,l}$ by $D_{n,l}$, and denote by

$$D_n = D_{n,1} \oplus \cdots \oplus D_{n,l_n} \oplus B_{n,l_n+1} \oplus \cdots \oplus B_{n,k_n}.$$

It is clear that

$$K_0(D_n) \cong K_0(B_n) \quad \text{and} \quad K_1(D_n) \cong \bigoplus_{l=1}^{l_n} \mathbb{Z}/p_{n,l}\mathbb{Z} \cong G_n$$

Let us construct maps $\chi_n : D_n \to D_{n+1}$ as the following.

For the direct summand $D_{n,i}$ and any direct summand $D_{n+1,j}$, if $p_{n,i}$ and $p_{n+1,j}$ are non-zero, by Corollary 3.9 of [11], there is a map $\chi_n(i,j) : D_{n+1,i} \to D_{n,j}$ such that

$$[\chi_n(i,j)]_0 = [\varphi_n(i,j)]_0 = r_{n,i,j}$$

and

$$[\chi_n(i,j)]_1 = \iota_n(i,j).$$

If $p_{n,i} = 0$ and $p_{n+1,j} \neq 0$, then define the map

$$\chi_n(i,j) : \mathrm{M}_{m_{n,i}}(\mathbb{C}) \otimes (\mathrm{C}(\mathbb{T})) \cong D_{n,i} \to pD_{n+1,j}p \cong \mathrm{M}_{r_{n,i,j}m_{n,i}}(\mathbb{C}) \otimes \mathbf{I}[1, p_{n+1,j}, 1]$$

where $p$ is a projection stands for $r_{n,i,j}m_{n,i} \in K_0(D_{n+1,j})$, by

$$e \otimes z \to \mathrm{diag}\{e \otimes u, e \otimes z_1, ..., e \otimes z_{r_{n,i,j}-1}\},$$

where $z$ is the standard unitary $z \mapsto z$, $z_i$ are certain points in the unit circle, and $u$ is a unitary in $pD_{n+1,j}p$ which represents $\iota_n(i,j)(1)$. Then, it is clear that

$$[\chi_n(i,j)]_0 = [\varphi_n(i,j)]_0 = r_{n,i,j}$$

and

$$[\chi_n(i,j)]_1 = \iota_n(i,j).$$

A similar argument for $p_{n,i} \neq 0$ and $p_{n+1,j} = 0$ also provides a homomorphism $\chi_n(i,j)$ which induces the right $K$-theory map. Moreover, the argument above also applies to the maps between $D_{n,i}$ and $B_{n+1,j}$, and between $B_{n,i}$ and $D_{n+1,j}$, such that there is a map $\chi_n(i,j)$ with

$$[\chi_n(i,j)]_0 = [\varphi_n(i,j)]_0 = r_{n,i,j}$$

and

$$[\chi_n(i,j)]_1 = \iota_n(i,j).$$

For direct summand $B_{n,i}$ and $B_{n+1,j}$, define

$$\chi_n(i,j) = \varphi_n(i,j).$$

In this way, we have a homomorphism $\chi_n : D_n \to D_{n+1}$ satisfying

$$[\chi_n]_0 = [\varphi_n] \quad \text{and} \quad [\chi_n]_1 = \iota_{n,n+1}$$

Let us consider the inductive limit

$$D = \varinjlim_{n \to \infty} (D_n, \chi_n).$$

It is clear that $K_0(D) = K_0(B)$ and $K_1(D) = \varinjlim(G_n, \iota_n) = K_0(A)$. With a suitable choice of $\chi_n(i,j)$ between $D_{n,i}$ and $D_{n+1,j}$, $D_{n,i}$ and $B_{n+1,j}$, and $B_{n,i}$ and $D_{n+1,j}$, we may assume that $D$ is a simple C*-algebra. Let us show $\mathrm{TR}(D) \leq 1$.

From the construction, it is clear that $D$ has the following property: For any finite subset $\mathcal{F} \subset D$ and any $\epsilon > 0$, there exists $n$ such that if denote by $I_n = \bigoplus_{i=l_n+1}^{p_n} B_{n,i}$ and $p_n = 1_{I_n}$, then, for any $x \in \mathcal{F}$

(1) $||[p_n, x]|| \le \epsilon$, and

(2) there is $a \in I_n$ such that $||p_n x p_n - a|| \le \epsilon$.

By Theorem 3.2 of [17], the C*-algebra has the property (SP), that is, any nonzero hereditary sub-C*-algebra contains a nonzero projection. Thus, in order to show $\mathrm{TR}(D) \le 1$, one only has to show that for any given projection $q \in D$, one can choose $n$ sufficiently large such that $1 - p_n \lesssim q$.

Note that the C*-algebra $D$ is an inductive limit of dimension drop interval algebras together with circle algebras, which satisfy the strict comparison on projections, i.e., for any two projections $e$ and $f$, if $\tau(e) < \tau(f)$ for any tracial state $\tau$, then $e \lesssim f$. Then $D$ also has the strict comparison on projections. (See, for example, Theorem 4.12 of [8].)

Therefore, in order to show $1 - p_n \lesssim q$, one only has to show that for any given $\epsilon > 0$, there is a sufficiently large $n$ such that $\tau(1 - p_n) \le \epsilon$ for any $t \in T(D)$. However, this condition can be fulfilled by Equation e 4.2, and thus, the C*-algebra $D$ is tracial rank one.

Let us show that $B$ and $D$ has the same tracial simplex, and has the same pairing with the $K_0$-group. Consider the non-unital C*-algebra

$$C = \varinjlim_{n\to\infty} (\bigoplus_{i=l_n+1}^{k_n} B_{n,i}, \psi_n),$$

where the map $\psi_n$ is the restriction of $\varphi_n$ to $\bigoplus_{i=l_n+1}^{k_n} B_{n,i}$ and $\bigoplus_{i=l_{n+1}+1}^{k_{n+1}} B_{n+1,i}$. Then, by Lemma 10.8 (and its proof) of [17], there are isomorphisms $r^{\#}$ and $r_{\#}$, and $s^{\#}$ and $s_{\#}$ such that

$$\begin{array}{ccc} T(B) & \xrightarrow{r_B} & S_u(K_0(B)) \\ \downarrow{\scriptstyle r^{\#}} & & \uparrow{\scriptstyle r_{\#}} \\ T(C) & \xrightarrow{r_C} & S_{u'}(K_0(C)), \end{array} \qquad \begin{array}{ccc} T(D) & \xrightarrow{r_D} & S_u(K_0(D)) \\ \downarrow{\scriptstyle s^{\#}} & & \uparrow{\scriptstyle s_{\#}} \\ T(C) & \xrightarrow{r_C} & S_{u'}(K_0(C)) \end{array}$$

commutes. Therefore, there are isomorphisms $t^{\#}$ and $t_{\#}$ such that

$$\begin{array}{ccc} T(B) & \xrightarrow{r_B} & S_u(K_0(B)) \\ \downarrow{\scriptstyle t^{\#}} & & \uparrow{\scriptstyle t_{\#}} \\ T(D) & \xrightarrow{r_D} & S_{u'}(K_0(D)) \end{array}$$

commutes. Therefore, the C*-algebra $B$ and $D$ has the same simplex of traces and pairing map. Hence,

$$((K_0(D), K_0(D)_+, [1_D]), T(D), \lambda_A) \cong ((K_0(B), K_0(B)_+, [1_B]), T(B), \lambda_B),$$

and therefore by Equation e 4.1,

$$((K_0(D), K_0(D)_+, [1_D]), K_1(D), T(D), \lambda_A) \cong ((K_0(A), K_0(A)_+, [1_A]), K_1(A)T(A), \lambda_A).$$

Since $D$ is also an inductive limit of dimension drop interval algebras together with circle algebras, by (the proof of) Theorem 9.9 of [19], one has that $D$ is a simple inductive limit of dimension drop circle algebra, and hence by Theorem 11.7 of [19], one has that $A \cong D$. Since $TR(D) \le 1$, we have that $TR(A) \le 1$, as desired. □

**Theorem 4.2.** *Every unital simple $A\mathbb{T}D$-algebra is in $\mathcal{A}_{\mathcal{Z}}$.*

*Proof.* If $K_0(A)/\mathrm{Inf}(K_0(A)) \ncong \mathbb{Z}$, then, 4.1 shows that $TR(A) \leq 1$. By Theorem 10.4 of [17], $A$ is in fact a unital simple AH-algebra with no dimension growth. It follows from [6] that $A$ is also approximately divisible. It follows from Theorem 2.3 of [27] that $A$ is $\mathcal{Z}$-stable. So $A \in \mathcal{A}_{\mathcal{Z}}$.

Let $A$ be a general unital simple A$\mathbb{T}$D-algebra. Let $\mathfrak{p}$ be a supernatural number of infinite type. Then $K_0(A \otimes M_{\mathfrak{p}})/\ker\rho_A \ncong \mathbb{Z}$. It follows from 4.1 that $TR(A \otimes M_{\mathfrak{p}}) \leq 1$. Thus $A \in \mathcal{A}$.

It follows from Theorem 4.5 of [27] that $A$ is $\mathcal{Z}$-stable. Therefore $A \in \mathcal{A}_{\mathcal{Z}}$. □

**4.3.** From Theorem 4.2 we see that Theorem 2.13 also unifies the classification theorems of [17] and that of [19]. In the next two sessions, we will show that $\mathcal{A}$ contains many more $C^*$-algebras.

# 5 Rationally Riesz Groups

**Theorem 5.1.** *For any countable abelian groups $G_{00}$ and $G_1$, any group extension:*

$$0 \to G_{00} \to G_0 \xrightarrow{\pi} \mathbb{Z} \to 0,$$

*with*

$$(G_0)_+ = \{x \in G_0 : \pi(x) > 0 \text{ and } x = 0\},$$

*any order unit $u \in G_0$ and any metrizable Choquet simple $S$, there exists a unital simple ASH-algebra $A \in \mathcal{A}$ such that*

$$((K_0(A), K_0(A)_+, [1_A]), K_1(A), T(A)) = ((G_0, (G_0)_+, u), G_1, S).$$

*Proof.* Let $S_0$ be the point which corresponding to the unique state on $G_0$. It follows a theorem of Elliott ([5]) that there exists a unital simple ASH-algebra $B$ such that

$$((K_0(B), K_0(B)_+, [1_B]), K_1(B), T(B)) = ((G_0, (G_0)_+, u), G_1, S_0).$$

Then $B \otimes M_{\mathfrak{p}}$ is a unital separable simple $C^*$-algebra which is approximately divisible and the projections of $B \otimes M_{\mathfrak{p}}$ separate the tracial state space (in this case it contains a single point). Thus $B \otimes M_{\mathfrak{p}}$ has real rank zero and stable rank one by [22]. Thus, by Proposition 5.4 of [15], $TR(B \otimes M_{\mathfrak{p}}) = 0$. Let $B_0$ be the unital simple A$\mathbb{T}$D-algebra (see Theorem 4.5 of [11]) such that

$$((K_0(B_0), K_0(B)_+, [1_{B_0}]), K_1(B_0), T(B_0), \lambda_{B_0}) = ((\mathbb{Z}, \mathbb{N}, 1), \{0\}, S, r_S).$$

It follows from 4.2 that $B_0 \in \mathcal{A}$. By Theorem 11.10 (iv) of [18], $B_0 \otimes B \in \mathcal{A}$. Define $A = B_0 \otimes B$. One then calculates that

$$((K_0(A), K_0(A)_+, [1_A]), K_1(A), T(A)) = ((G_0, (G_0)_+, u), G_1, S).$$

□

**Theorem 5.2.** *For any countable weakly unperforated simple Riesz group $G_0$ with order unit $u$, any countable abelian group $G_1$ and any metrizable Choquet simplex $S$ and any surjective homomorphism $r_S : S \to S_u(G_0)$ which maps $\partial_e(S)$ onto $\partial_e(S_u(G_0))$. There exists a unital simple ASH-algebra $A \in \mathcal{A}_z$ such that*

$$((K_0(A), K_0(A)_+, [1_A]), K_1(A), T(A), \lambda_A) = ((G_0, (G_0)_+, u), G_1, S, r_S).$$

*Proof.* First, we assume that $G_0/\mathrm{Inf}(G_0) \ncong \mathbb{Z}$. It follows from a theorem of Villadsen ([29]) that, in this case, there is a unital simple AH-algebra $C$ with no dimension growth (and with tracial rank no more than one—see Theorem 2.5 of [17]) such that

$$((K_0(C), K_0(C)_+, [1_C]), K_1(C), T(C), \lambda_C) = ((G_0, (G_0)_+, u), G_1, S, r_S).$$

The case that $G_0/\mathrm{Inf}(G_0) \cong \mathbb{Z}$ follows from 5.1. □

**Remark 5.3.** Theorem 5.2 includes all unital simple A𝕋D-algebras. Let $A$ be a unital simple $C^*$-algebra in $\mathcal{A}$ with weakly unperforated Riesz group $K_0(A)$. If $K_0(A)/\mathrm{Inf}(K_0(A)) \ncong \mathbb{Z}$, then $TR(A) \leq 1$. By the classification result in [17], $A$ is in fact a unital simple AH-algebra. If $K_0(A) = \mathbb{Z}$, then, by 4.2, $A$ is a unital simple A𝕋D-algebra. However, Theorem 5.2 contains unital simple $C^*$-algebra $A$ for which $K_1(A)$ is an arbitrary countable abelian group. These $C^*$-algebras can not be isomorphic to a unital simple A𝕋D-algebras (see Theorem 1.4 of [19]).

**Definition 5.4.** Let $G$ be a partially ordered group. We say $G$ has rationally Riesz property if the following holds: For two pairs of elements $x_1, x_2$, and $y_1, y_2 \in G$ with $x_i \leq y_j$, $i, j = 1, 2$, there exists $z \in G$ such that

$$nz \leq my_i \text{ and } mx_i \leq nz, \;\; i = 1, 2, .$$

where $m, n \in \mathbb{N} \setminus \{0\}$.

Let $G$ and $H$ be two weakly unperforated simple ordered groups with order-unit $u$ and $v$ respectively. Consider the group $G \otimes H$. Set the semigroup

$$(G \otimes H)_+ = \{a \in G \otimes H;\; (s_1 \otimes s_2)(a) > 0, \forall s_1 \in S_u(G), \forall s_2 \in S_v(H)\} \cup \{0\}.$$

Since $G_+ \otimes H_+ \subseteq (G \otimes H)_+$ and $G \otimes H = G_+ \otimes H_+ - G_+ \otimes H_+$, one has

$$G \otimes H = (G \otimes H)_+ - (G \otimes H)_+.$$

For any $a \in ((G \otimes H)_+) \cap (-(G \otimes H)_+)$, if $a \neq 0$, then $(s_1 \otimes s_2)(a) > 0$ for any $s_1 \in S_u(G)$ and $s_2 \in S_v(H)$, and $(s_1 \otimes s_2)(-a) > 0$, for any $s_1 \in S_u(G)$ and $s_2 \in S_v(H)$, which is a contradiction. Therefore,

$$((G \otimes H)_+) \cap (-(G \otimes H)_+) = \{0\}.$$

Moreover, since $(s_1 \otimes s_2)(u \otimes v) = 1$ for any $s_1 \in S_u(G)$ and $s_2 \in S_v(H)$, and $S_u(G)$ and $S_v(H)$ are compact, for any element $a \in (G \otimes H)$, there is a natural number $m$ such that

$$m(u \otimes v) - a \in (G \otimes H)_+.$$

Hence $(G \otimes H, (G \otimes H)_+, u \otimes v)$ is a scaled ordered group.

**Lemma 5.5.** *Let $G$ and $H$ be simple ordered groups with order units $u$ and $v$ respectively. If $H$ has a unique state $\tau$, then for any state $s$ on the ordered group $G \otimes H$, one has*

$$s(g \otimes h) = s(g \otimes v)\tau(h)$$

*for any $g \in G$, $h \in H$.*

*Proof.* It is enough to show the statement for strictly positive $g$ and $h$. For each $g \in G_+ \setminus \{0\}$, since $G$ is simple, one has that $u \leq mg$ for some natural number $m$. Hence

$$s(g \otimes v) \geq \frac{1}{m} s(u \otimes v) = \frac{1}{m},$$

and in particular, $s(g \otimes v) \neq 0$. Consider the map $s_g : H \to \mathbb{R}$ defined by

$$s_g(h) = \frac{s(g \otimes h)}{s(g \otimes v)}.$$

Then $s_g$ is a state of $H$. Since $\tau$ is the unique state of $H$. One has that $s_g = \tau$, and hence

$$\frac{s(g \otimes h)}{s(g \otimes v)} = \tau(h) \quad \text{for any } h \in H.$$

Therefore,

$$s(g \otimes h) = s(g \otimes v)\tau(h),$$

as desired. □

**Lemma 5.6.** *Let $G$ be a countable weakly unperforated simple partially ordered group with an order unit $u$. Then, for any dense subgroup $D$ of $\mathbb{R}$ containing 1, the map $\lambda : S_{u\otimes 1}(G\otimes D) \to S_u(G)$ defined by*

$$\lambda(s)(x) = s(j(x)) \text{ for all } x \in G,$$

*where $j : G \to G\otimes D$ defined by $j(x) = x\otimes 1$ for all $x$, is an affine homeomorphism.*

*Proof.* Since any dense subgroup of $\mathbb{R}$ with the induced order has unique state, the statement follows from Lemma 5.5 directly. □

**Proposition 5.7.** *Let $G$ be a countable weakly unperforated simple partially ordered group with an order unit $u$. Then the following are equivalent:*

(1) *$G$ has the rationally Riesz property;*

(2) *$G\otimes D$ has the Riesz property for some dense subgroup $D$ of $\mathbb{R}$ containing* 1;

(3) *$G\otimes D$ has the Riesz property for all dense subgroups $D$ of $\mathbb{R}$ containing* 1;

(4) *For any two pairs of elements $x_i$ and $y_i \in G$ with $x_i \le y_j$, $i, j = 1, 2$, there is an element $z \in G$ and a real number $r > 0$ such that*

$$s(x_i) \le rs(z) \le s(y_j), \text{ for all } s \in S_u(G),\ i, j = 1, 2.$$

*Proof.* It is clear, by the assumption that $G$ is weakly unperforated, that (1) $\Longrightarrow$ (2) for $D = \mathbb{Q}$.

It is also clear that (3) $\Longrightarrow$ (2).

That (2) $\Longrightarrow$ (4) follows from 5.6.

Suppose that (4) holds. Let $x_i \le y_j$ be in $G$, $i, j = 1, 2$. If one of $x_i$ is the same as one of $y_j$, say $x_1 = y_1$, then (2) holds for $z = x_2$ and $m = n = 1$. Thus, let us assume that $x_i < y_j$, $i, j = 1, 2$. Since $G$ is simple, $S_u(G)$ is compact. It follows from (4) there is a rational number $r \in \mathbb{Q}$ such that

$$s(x_i) < rs(z) < s(y_j), \text{ for all } s \in S_u(G),\ i, j = 1, 2.$$

Write $r = n/m$ for some $m, n \in \mathbb{N}$. Then

$$s(mx_i) < s(nz) < s(my_j) \text{ for all } s \in S_u(G),\ i, j = 1, 2.$$

It follows from Theorem 6.8.5 of [1] that

$$mx_i \le nz \le my_j \ \ i, j = 1, 2.$$

Thus (4) $\Longrightarrow$ (1).

By applying 5.6, it is even easier to shows that (4) $\Longrightarrow$ (3). □

**Proposition 5.8.** *Let $G$ be a countable weakly unperforated simple partially ordered group with an order unit $u$. Then $G$ has the rationally Riesz property if and only if $S_u(G)$ is a metrizable Choquet simplex.*

*Proof.* Suppose that $G$ has the rationally Riesz property. Then, by 5.7, $G\otimes\mathbb{Q}$ is a weakly unperforated simple Riesz group and $(G\otimes\mathbb{Q})/\mathrm{Inf}(G\otimes\mathbb{Q}) \not\cong \mathbb{Z}$. Put $F = (G\otimes\mathbb{Q})/\mathrm{Inf}(G\otimes\mathbb{Q})$. Then, it follows that $F$ is a simple dimension group. It then follows from the Effros-Handelman-Shen Theorem ([3]) that there exists a unital simple AF-algebra $A$ with

$$(K_0(A), K_0(A), [1_A]) = (F, F_+, \bar{u}),$$

where $\bar{u}$ is the image of $u$ in $F$. It follows that $T(A) = S_u(F)$. By Theorem 3.1.18 of [25], $T(A)$ is a metrizable Choquet simplex. It follows from 5.6 that $S_u(G)$ is a metrizable Choquet simplex.

For the converse, let $G$ be a countable weakly unperforated simple partially ordered group with an order unit $u$ so that $S_u(G)$ is a metrizable Choquet simplex. Let $F = G \otimes \mathbb{Q}$. By 5.7, it suffices to show that $F$ has the Riesz property. By 5.6, $S_{u\otimes 1}(F)$ is a metrizable Choquet simplex. It follows from Theorem 11.4 of [10] that $\mathrm{Aff}(S_{u\otimes 1}(F))$ has the Riesz property. Let $\rho : F \to \mathrm{Aff}(S_{u\otimes 1}(F))$ be the homomorphism defined by

$$\rho(g)(s) = s(g) \text{ for all } s \in S_{u\otimes 1}(G) \text{ and } \text{ for all } g \in F.$$

Define $F_1 = F + \mathbb{R}(u \otimes 1)$ and extend $\rho$ from $F_1$ into $\mathrm{Aff}(S_{u\otimes 1}(F))$ in an obvious way. Then $\rho(F_1)$ contains the constant functions. It also separates the points. By Corollary 7.4 of [10], the linear space generated by $\rho(F_1)$ is dense in $\mathrm{Aff}(S_{u\otimes 1}(G))$.

Moreover, for any real number $r < 1$ and any positive element $p \in F$, the positive affine function $r\rho(p)$ can be approximated by elements in $\rho(F)$. It follows that $\rho(F)$ is dense in $\mathrm{Aff}(S_{u\otimes 1}(G))$. It follows that $\rho(F)$ has the Riesz property. Moreover, since the order of $F$ is given by $\rho(F)$ (see Theorem 6.8.5 of [1]), this implies that $F$ has the Riesz property. □

**Example 5.9.** Let $H$ be any nontrivial group. Then ordered group $\mathbb{Z} \oplus H$ with the order induced by $\mathbb{Z}$ is a simple ordered group, which satisfies the rationally Riesz property. However, it is not a Riesz group.

Let $\Gamma$ be a cardinality at most countable and bigger than 1. Then, the ordered group $G = \bigoplus_\Gamma \mathbb{Z}$ with the positive cone $\{0\} \cup \bigoplus_\Gamma \mathbb{Z}^+$ is simple, since any positive element is an order unit. Consider $a_1 = (1, 0, 0, 0, ...)$, $a_2 = (0, 1, 0, 0, ...)$, $a_3 = (2, 2, 0, 0, ...)$, and $a_4 = (2, 3, 0, 0, ...)$. Then $a_1, a_2 \le a_3, a_4$. However, one can not find an element $b$ such that $a_1, a_2 \le b \le a_3, a_4$. Hence, the group $G$ is not Riesz. But this group has rationally Riesz property.

**Proposition 5.10.** *Let $A \in \mathcal{A}$ be a $\mathcal{Z}$-stable $C^*$-algebra. Then $(K_0(A), K_0(A)_+, [1_A])$ is a countable weakly unperforated simple partially ordered group with order unit $[1_A]$ which has the rationally Riesz property. Moreover $S_{[1_A]}(K_0(A))$ is a metrizable Choquet simplex.*

*Proof.* It follows from [9] that $(K_0(A), K_0(A)_+, [1_A])$ is a countable weakly unperforated simple partially ordered group with order unit $[1_A]$. Since $TR(A \otimes \mathbb{Q}) \le 1$, $(K_0(A \otimes \mathbb{Q}), K_0(A \otimes \mathbb{Q})_+, [1_{A\otimes\mathbb{Q}}])$ is a Riesz group. It follows that $(K_0(A), K_0(A)_+, [1_A])$ has the rationally Riesz property. By 5.6 and 5.8, $S_{[1_A]}(K_0(A))$ is a metrizable Choquet simplex.

□

**Lemma 5.11.** *Let $G_0$ be a countable weakly unperforated simple partially ordered group with order unit u which also has the rationally Riesz property and any countable abelian group $G_1$. There exists a unital simple ASH-algebra $A \in \mathcal{A}_{0z} \subset \mathcal{A}_z$ such that*

$$((K_0(A), K_0(A)_+, [1_A]), K_1(A), T(A)) = ((G_0, (G_0)_+, u), G_1, S_u(G_0)).$$

*Proof.* It follows from 5.8 that $S_u(G_0)$ is a Choquet simplex. It follows [5] that there exists a unital simple ASH-algebra $A$ such that

$$((K_0(A), K_0(A)_+, [1_A]), K_1(A), T(A)) = ((G_0, (G_0)_+, u), G_1, S_u(G_0)).$$

We may assume that $A \cong A \otimes \mathcal{Z}$. Note that the set of projections of $A \otimes M_{\mathfrak{p}}$ separate the tracial state space in this case for any supernatural number $\mathfrak{p}$. It follows from the argument of 8.2 of [30] that $TR(A \otimes M_{\mathfrak{p}}) = 0$. In particular, $A \in \mathcal{A}$. □

**Definition 5.12.** Let $T_1$ and $T_2$ be two finite simplexes with vertices $\{e_1, ..., e_m\}$ and $\{f_1, ..., f_n\}$. Let us denote by $T_1 \dot{\times} T_2$ the finite simplex spanned by vertices $(e_i, f_j)$, $1 \le i \le m$, $1 \le j \le n$.

**Lemma 5.13.** *Let $A$ be unital $C^*$-algebra and $\tau$ a tracial state of $A$. Then $\tau$ is extremal if and only if $\pi_\tau(A)''$ in the* ΓHS-*representation $(\pi_\tau, \mathscr{H}_\tau)$ is a* $\mathrm{II}_1$ *factor.*

*Proof.* Note that $\pi_\tau(A)''$ is always of type $\mathrm{II}_1$. Assume that $\tau$ is extremal. If $\pi_\tau(A)''$ were not a factor, then, there is a nontrivial central projection $p \in \pi_\tau(A)''$.

We claim that $\tau(p) \neq 0$. Suppose that $a_n \in A_+$ such that $\{\|\pi_\tau(a_n)\|\}$ is bounded and $\pi_\tau(a_n)$ converges to $p$ in the weak operator topology in $B(\mathscr{H}_\tau)$. If $\tau(p) = 0$, then $\tau(a_n) \to 0$. It follows that $\tau(a_n^2) \to 0$. For any $a \in A \setminus \{0\}$,

$$\tau(aa_na^*) = \tau(a_na^*a) \leq (\tau(a_n^2)\tau((a^*a)^2))^{1/2} \to 0.$$

It follows that

$$\lim_{n\to\infty} \langle \pi_\tau(a_n)\xi_a, \xi_a \rangle_{\mathscr{H}_\tau} = 0$$

for any $a \in A$, where $\xi_a$ is the vector given by $a$ in the ΓHS construction, which implies that $\pi_\tau(a_n) \to 0$ in the weak operator topology. Therefore $p = 0$. This proves the claim.

By the claim neither $\tau(p)$ nor $\tau(1-p)$ is zero. Thus $0 < \tau(p) < 1$. Define $\tau_1 : a \mapsto \frac{1}{\tau(p)}\tau(pap)$ and $\tau_2 : a \mapsto \frac{1}{\tau(1-p)}\tau((1-p)a(1-p))$. Note that $\tau_1(p) = 1$. Thus $\tau_1 \neq \tau$.

Then $\tau_1$ and $\tau_2$ are traces on $\pi_\tau(A)''$, and $\tau(a) = \tau(p)\tau_1(a) + \tau(1-p)\tau_2(a)$ for any $a \in \pi_\tau(A)''$. Therefore, the trace $\tau$ on $\pi_\tau(A)''$ is not extremal. Since both $\tau_1$ and $\tau_2$ are also normal, we conclude that $\tau$ is also not extremal on $A$, which contradicts to the assumption.

Conversely, assume that $\pi_\tau(A)''$ is a factor. If $\tau = \lambda\tau_1 + (1-\lambda)\tau_2$ with $\lambda \in (0,1)$, then it is easy to check that $\tau_1$ and $\tau_2$ can be extended to normal states on $\pi_\tau(A)''$, and hence to traces on $\pi_\tau(A)''$. Therefore $\tau_1 = \tau_2$, and $\tau$ is extremal. □

**Lemma 5.14.** *Let $A$ and $B$ be two unital $C^*$-algebras. Let $\tau$ be an extremal tracial state on a $C^*$-algebra tensor product $A \otimes B$. Then, the restriction of $\tau$ to $A$ or $B$ is an extremal trace.*

*Proof.* Denote by the restriction of $\tau$ to $A$ and $B$ by $\tau_A$ and $\tau_B$. Consider the ΓHS-representation $(\pi_\tau, \mathscr{H}_\tau)$ of $A \otimes B$. Since $\tau$ is extremal, by Lemma 5.13, the von Neumann algebra $\pi_\tau(A \otimes B)''$ is a $\mathrm{II}_1$ factor. Since $(A \otimes 1)$ commutes with $(1 \otimes B)$, $\pi_\tau(A \otimes 1)''$ and $\pi_\tau(1 \otimes B)''$ are also $\mathrm{II}_1$ factors.

Set $p$ the orthogonal projection to the closure of the subspace spanned by $\{(a \otimes 1)(\xi); \xi \in \mathscr{H}_\tau, a \in A\}$. Then the ΓHS-representation $(\pi_{\tau_A}, \mathscr{H}_{\tau_A})$ of $A$ is unitarily equivalent to the cut-down of $\pi_\tau$ to $A$ and $p\mathscr{H}_\tau$. Hence $\pi_{\tau_A}(A)''$ is a $\mathrm{II}_1$ factor in $\mathcal{B}(\mathscr{H}_{\tau_A})$. By Lemma 5.13, $\tau_A$ is an extremal trace of $A$. The same argument works for $B$. □

**Lemma 5.15.** *Let $A$ and $B$ be two unital $C^*$-algebras and let $\tau$ be a tracial state of a $C^*$-algebra tensor product $A \otimes B$. Then, if the restriction of $\tau$ to $B$ is an extremal trace, one has that $\tau(a \otimes b) = \tau(a \otimes 1)\tau(1 \otimes b)$ for all $a \in A$ and $b \in B$.*

*Proof.* We may assume that $a$ is a positive element with norm one. If $\tau(a \otimes 1) = 0$, then $\tau(a \otimes b) = 0$ by Cauchy-Schwartz inequality, and equation holds. If $\tau(a \otimes 1) = 1$, then the equation also hold by considering the element $(1-a) \otimes 1$.

Therefore, we may assume that $\tau(a \otimes 1) \neq 0, 1$. Fix $a$. Then, we have

$$\tau(1 \otimes b) = \tau(a \otimes 1)\frac{\tau(a \otimes b)}{\tau(a \otimes 1)} + (1 - \tau(a \otimes 1))\frac{\tau((1-a) \otimes b)}{1 - \tau(a \otimes 1)} \quad \text{for any } b \in B.$$

Note that both $b \mapsto \frac{\tau(a \otimes b)}{\tau(a \otimes 1)}$ and $b \mapsto \frac{\tau((1-a)\otimes b)}{1-\tau(a\otimes 1)}$ are tracial states of $B$. Since $b \mapsto \tau(1 \otimes b)$ is an extremal trace, one has that $\frac{\tau(a\otimes b)}{\tau(a\otimes 1)} = \tau(1 \otimes b)$ for any $b \in B$. Therefore, the equation

$$\tau(a \otimes b) = \tau(a \otimes 1)\tau(1 \otimes b)$$

holds for any $a \in A$ and $b \in B$. □

**Corollary 5.16.** *Let $A$ and $B$ be two unital $C^*$-algebras and $\tau$ an extremal tracial state of a $C^*$-algebra tensor product $A \otimes B$. Then $\tau$ is the product of its restrictions to $A$ and $B$*

*Proof.* It follows from Lemma 5.14 and Lemma 5.15. □

**Corollary 5.17.** *Let $A$ and $B$ be two $C^*$-algebras with simplexes of traces $\mathrm{T}(A)$ and $\mathrm{T}(B)$. If $\mathrm{T}(A)$ and $\mathrm{T}(B)$ have finitely many extreme points, then $T(A \otimes B) = \mathrm{T}(A) \dot{\times} \mathrm{T}(B)$.*

*Proof.* It follows from Corollary 5.16 directly. □

**Theorem 5.18.** *Let $G_0$ be a countable weakly unperforated simple partially ordered group with an order unit $u$ which has the rationally Riesz property, let $G_1$ be a countable abelian group, and let $T$ be any finite simplex. Assume that $S_u(G_0)$ has only finitely many extreme points. Then there exists a unital simple ASH-algebra $A \in \mathcal{A}$ such that*

$$((K_0(A), K_0(A)_+, [1_A]), K_1(A), T(A), \lambda_A) = ((G_0, (G_0)_+, u), G_1, T \dot{\times} S_u(G_0), r),$$

*where $r : T \dot{\times} S_u(G_0) \to S_u(G_0)$ is defined by $(\tau, s)(x) = s(x)$ for all $x \in G_0$ and for all extremal tracial state $\tau \in T$ and extremal state $s \in S_u(G_0)$.*

*Proof.* From 5.11, there exists a unital simple ASH-algebra $B \in \mathcal{A}$ such that

$$((K_0(B), K_0(B)_+, [1_B]), K_1(B), T(B)) = ((G_0, (G_0)_+, u), G_1, S_u(G_0)).$$

Then let $B_0$ be a unital simple A$\mathbb{T}$D-algebra with

$$((K_0(B_0), K_0(B_0)_+, [1_{B_0}]), K_1(B_0), T(B_0)) = ((\mathbb{Z}, \mathbb{N}, 1), \{0\}, T).$$

Define $A = B_0 \otimes B$. Then $A \in \mathcal{A}$. One checks that

$$((K_0(A), K_0(A)_+, [1_A]), K_1(A), T(A), \lambda_A) = ((G_0, (G_0)_+, u, G_1), T \dot{\times} S_u(G_0), r).$$

□

# 6 The range

Suppose that $A$ and $B$ are two stably finite unital $\mathcal{Z}$-stable $C^*$-algebras and suppose that there is a homomorphism

$$\Lambda : \mathrm{Ell}(A) \to \mathrm{Ell}(B).$$

There is $\Lambda_{\mathfrak{p}} : \mathrm{Ell}(A \otimes M_{\mathfrak{p}}) \to \mathrm{Ell}(B \otimes M_{\mathfrak{p}})$ and $\Lambda_{\mathfrak{q}} : \mathrm{Ell}(A \otimes M_{\mathfrak{p}}) \to \mathrm{Ell}(B \otimes M_{\mathfrak{q}})$ induced by $\Lambda$ so that the following diagram commutes

$$\begin{array}{ccccc}
\mathrm{Ell}(A \otimes M_{\mathfrak{p}}) & \stackrel{(\mathrm{id}_A \otimes 1)_*}{\longleftarrow} & \mathrm{Ell}(A) & \stackrel{(\mathrm{id}_A \otimes 1)_*}{\longrightarrow} & \mathrm{Ell}(A \otimes M_{\mathfrak{q}}) \\
\downarrow_{\Lambda_{\mathfrak{p}}} & & \downarrow_{\Lambda} & & \downarrow_{\Lambda_{\mathfrak{q}}} \\
\mathrm{Ell}(B \otimes M_{\mathfrak{p}}) & \stackrel{(\mathrm{id}_B \otimes 1)_*}{\longleftarrow} & \mathrm{Ell}(B) & \stackrel{(\mathrm{id}_B \otimes 1)_*}{\longrightarrow} & \mathrm{Ell}(B \otimes M_{\mathfrak{q}})
\end{array}$$

**Lemma 6.1.** *Let $A$ and $B$ be two $\mathcal{Z}$-stable $C^*$-algebras in $\mathcal{A}$ and let $\mathfrak{p}$ and $\mathfrak{q}$ be two supernatural numbers of infinite type which are relatively prime. Suppose that*

$$\Lambda : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$$

*is a homomorphism. Then there is a unitarily suspended $C([0,1])$-unital homomorphisms $\varphi : A\otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to B\otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ such that $Ell(\pi_0\circ\varphi)=\Lambda_{\mathfrak{p}}$ and $Ell(\pi_1\circ\varphi)=\Lambda_{\mathfrak{q}}$ so that the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{Ell}(A\otimes M_{\mathfrak{p}}) & \stackrel{(\mathrm{id}_A\otimes 1)_*}{\leftarrow} & \mathrm{Ell}(A) & \stackrel{(\mathrm{id}_A\otimes 1)_*}{\rightarrow} & \mathrm{Ell}(A\otimes M_{\mathfrak{q}}) \\ \downarrow_{(\pi_0\circ\varphi)_*} & & \downarrow_{\Lambda} & & \downarrow_{(\pi_1\circ\varphi)_*} \\ \mathrm{Ell}(B\otimes M_{\mathfrak{p}}) & \stackrel{(\mathrm{id}_B\otimes 1)_*}{\leftarrow} & \mathrm{Ell}(B) & \stackrel{(\mathrm{id}_B\otimes 1)_*}{\rightarrow} & \mathrm{Ell}(B\otimes M_{\mathfrak{q}}) \end{array}$$

*Proof.* Since $A,\, B\in\mathcal{A}$, $TR(A\otimes M_{\mathfrak{p}})\le 1$, $TR(A\otimes M_{\mathfrak{q}})\le 1$, $TR(B\otimes M_{\mathfrak{p}})\le 1$ and $TR(B\otimes M_{\mathfrak{q}})\le 1$, there is a unital homomorphism $\varphi_{\mathfrak{p}} : A\otimes M_{\mathfrak{p}}\to B\otimes M_{\mathfrak{p}}$ and $\psi_{\mathfrak{q}} : A\otimes M_{\mathfrak{q}}\to B\otimes M_{\mathfrak{q}}$ such that

$$\mathrm{Ell}(\varphi_{\mathfrak{p}})=\Lambda_{\mathfrak{p}}\ \text{ and }\ \mathrm{Ell}(\psi_{\mathfrak{q}})=\Lambda_{\mathfrak{q}}.$$

Put $\varphi=\varphi_{\mathfrak{p}}\otimes \mathrm{id}_{M_{\mathfrak{q}}} : A\otimes Q\to B\otimes Q$ and $\psi=\psi_{\mathfrak{q}}\otimes \mathrm{id}_{M_{\mathfrak{q}}} : A\otimes Q\to B\otimes Q$.

Note that

$$(\varphi)_{*i}=(\psi)_{*i}\ (i=0,1)\ \text{ and }\ \varphi_T=\psi_T.$$

(they are induced by $\Gamma$). Note that $\varphi_T$ and $\psi_T$ are affine homeomorphisms. Since $K_{*i}(B\otimes Q)$ is divisible, we in fact have $[\varphi]=[\psi]$ (in $KK(A\otimes Q, B\otimes Q)$). It follows from Lemma 11.4 of [18] that there is an automorphism $\beta : B\otimes Q\to B\otimes Q$ such that

$$[\beta]=[\mathrm{id}_{B\otimes Q}]\ \ KK(B\otimes Q, B\otimes Q)$$

such that $\varphi$ and $\beta\circ\psi$ are asymptotically unitarily equivalent. Since $K_1(B\otimes Q)$ is divisible, $H_1(K_0(A\otimes Q), K_1(B\otimes Q))=K_1(B\otimes Q)$. It follows that $\varphi$ and $\beta\circ\psi$ are strongly asymptotically unitarily equivalent. Note also in this case

$$\beta_T=(\mathrm{id}_{B\otimes Q})_T.$$

Let $\imath : B\otimes M_{\mathfrak{q}}\to B\otimes Q$ defined by $\imath(b)=b\otimes 1$ for $b\in B$. We consider the pair $\beta\circ\imath\circ\varphi_{\mathfrak{q}}$ and $\imath\circ\varphi_{\mathfrak{q}}$. By applying 11.5 of [18], there exists an automorphism $\alpha : \varphi_{\mathfrak{q}}(A\otimes M_{\mathfrak{q}})\to\varphi_{\mathfrak{q}}(A\otimes M_{\mathfrak{q}})$ such that $\imath\circ\alpha\circ\psi_{\mathfrak{q}}$ and $\beta\circ\imath\circ\psi_{\mathfrak{q}}$ are asymptotically unitarily equivalent (in $M(B\otimes Q)$). So they are strongly asymptotically unitarily equivalent. Moreover,

$$[\alpha]=[\mathrm{id}_{B\otimes M_{\mathfrak{q}}}]\ \text{ in }\ KK(B\otimes M_{\mathfrak{q}}, B\otimes M_{\mathfrak{q}}).$$

We will show that $\beta\circ\psi$ and $\alpha\circ\varphi_{\mathfrak{q}}\otimes\mathrm{id}_{M_{\mathfrak{p}}}$ are strongly asymptotically unitarily equivalent. Define $\beta_1=\beta\circ\imath\circ\psi_{\mathfrak{q}}\otimes\mathrm{id}_{M_{\mathfrak{p}}} : B\otimes Q\otimes M_{\mathfrak{p}}\to B\otimes Q\otimes M_{\mathfrak{p}}$. Let $j : Q\to Q\otimes M_{\mathfrak{p}}$ defined by $j(b)=b\otimes 1$. There is an isomorphism $s : M_{\mathfrak{p}}\to M_{\mathfrak{p}}\otimes M_{\mathfrak{p}}$ with $(\mathrm{id}_{M_{\mathfrak{q}}}\otimes s)_{*0}=j_{*0}$. In this case $[\mathrm{id}_{M_{\mathfrak{q}}}\otimes s]=[j]$. Since $K_1(M_{\mathfrak{p}})=0$. By 7.2 of [18], $\mathrm{id}_{M_{\mathfrak{q}}}\otimes s$ is strongly asymptotically unitarily equivalent to $j$. It follows that $\alpha\circ\psi_{\mathfrak{q}}\otimes\mathrm{id}_{M_{\mathfrak{p}}}$ and $\beta\circ\imath\circ\psi_{\mathfrak{q}}\otimes\mathrm{id}_{M_{\mathfrak{p}}}$ are strongly asymptotically unitarily equivalent. Consider the $C^*$-subalgebra $C=\beta\circ\psi(1\otimes M_{\mathfrak{p}})\otimes M_{\mathfrak{p}}\subset B\otimes Q\otimes M_{\mathfrak{p}}$. In $C$, $\beta\circ\varphi|_{1\otimes M_{\mathfrak{p}}}$ and $j_0$ are strongly asymptotically unitarily equivalent, where $j_0 : M_{\mathfrak{p}}\to C$ by $j_0(a)=1\otimes a$ for all $a\in M_{\mathfrak{p}}$. There exists a continuous path of unitaries $\{v(t) : t\in[0,\infty)\}\subset C$ such that

$$\lim_{t\to\infty}\mathrm{ad}\, v(t)\circ\beta\circ\varphi(1\otimes a)=1\otimes a\ \text{ for all }\ a\in M_{\mathfrak{p}}. \tag{e 6.3}$$

It follows that $\beta\circ\psi$ and $\beta_1$ are strongly asymptotically unitarily equivalent. Therefore $\beta\circ\psi$ and $\alpha\circ\psi_{\mathfrak{q}}\otimes\mathrm{id}_{M_{\mathfrak{p}}}$ are strongly asymptotically unitarily equivalent. Finally, we conclude that $\alpha\circ\psi_{\mathfrak{q}}\otimes\mathrm{id}_{\mathfrak{p}}$ and $\varphi$ are strongly asymptotically unitarily equivalent. Note that $\alpha\circ\psi_{\mathfrak{q}}$ is an isomorphism which induces $\Gamma_{\mathfrak{q}}$.

Let $\{u(t) : t \in [0, 1)\}$ be a continuous path of unitaries in $B \otimes Q$ with $u(0) = 1_{B\otimes Q}$ such that

$$\lim_{t\to\infty} \operatorname{ad} u(t) \circ \varphi(a) = \alpha \circ \psi_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{q}}}(a) \ \text{ for all } \ a \in A \otimes Q.$$

One then obtains a unitary suspended $C([0, 1])$-unital homomorphism which lifts $\Gamma$ along $Z_{p,q}$ (see [30]). □

**Theorem 6.2** (cf. Proposition 4.6 of [30]). *Let $A$ and $B$ be two $\mathcal{Z}$-stable $C^*$-algebra in $\mathcal{A}$. Suppose that there exists a strictly positive unital homomorphism*

$$\Lambda : \mathrm{Ell}(A) \to \mathrm{Ell}(B).$$

*Then there exists a unital homomorphism $\varphi : A \to B$ such that $\varphi$ induces $\Lambda$.*

*Proof.* The proof is a simple modification of that of Proposition 4.6 of [30] by applying 6.1.

First, it is clear that the proof of Lemma 4.3 of [30] holds if isomorphism is changed to homomorphism without any changes when both $A$ and $B$ are assumed to be simple. Moreover, the one-sided version of Proposition 4.4 also holds. In particular the part (ii) of that proposition holds. It follows that the homomorphism version of Proposition 4.5 of [30] holds since proof requires no changes except that we change the word "isomorphism" to "homomorphism" twice in the proof.

To prove this theorem, we apply 6.1 to obtain a unitarily suspended $C([0, 1])$-homomorphism $\varphi : A\otimes B\otimes \mathcal{Z}$ which has the properties described in 6.1. The rest of the proof is just a copy of the proof of Proposition 4.6 with only four changes: (1) $\tilde{\varphi} : A \otimes \mathcal{Z} \to B \otimes \mathcal{Z} \otimes \mathcal{Z}$ is a homomorphism (instead of isomorphism); (2) in the diagram (65), $\lambda$, $\lambda \otimes \mathrm{id}$ are homomorphisms (instead of isomorphism); (3) $\tilde{\varphi}_+$ is a homomorphism (instead of isomorphism); (4) since $\lambda$ and $\tilde{\varphi}_*$ agree as homomorphisms (instead of isomorphisms) and both are order homomorphisms preserving the order units, they also have to agree as such. □

**Definition 6.3.** A C*-algebra $A$ is said to be locally approximated by subhomogeneous C*-algebras if for any finite subset $\mathcal{F} \subseteq A$ and any $\epsilon > 0$, there is a C*-subalgebra $H \subseteq A$ isomorphic to a subhomogeneous algebra such that $\mathcal{F} \subseteq_\epsilon H$.

**Remark 6.4.** It is clear from the definition that any inductive limit of locally approximately subhomogeneous C*-algebras is again a locally approximately subhomogeneous C*-algebra.

**Lemma 6.5.** *Let that $(G_0, (G_0)_+, u)$ be a countable partially ordered weakly unperforated and rationally Riesz group, let $G_1$ be a countable abelian group, let $T$ be a metrizable Choquet simplex and let $\lambda_T : T \to S_u(G_0)$ be a surjective affine continuous map sending extremal points to extremal points. Suppose that $S_u(G_0)$ and $T$ have finitely many extremal points.*

*Then there exists one unital $\mathcal{Z}$-stable $C^*$-algebra $A \in \mathcal{A}$ such that*

$$\mathrm{Ell}(A) = ((G_0, (G_0)_+, [1_A]), G_1, T, \lambda_T).$$

*Moreover, the $C^*$-algebra $A$ can be locally approximated by subhomogeneous $C^*$-algebras.*

*Proof.* Denote by $e_1, e_2, ..., e_n$ the extreme points of $S_u(G_0)$, and denote by $S_1, ..., S_n$ the preimage of $e_1, ..., e_n$ under $\lambda$. Then, each $S_i$ is a face of $T$, and hence a simplex with finitely many extreme points. In each $S_i$, choose an extreme point $f_i$.

Set an affine map $\alpha : S_u(G_0)\dot{\times}T \to S_u(G_0)$ by

$$\alpha((e_i, g_j)) = e_i,$$

where $e_i$ is an extreme point of $S_u(G_0)$ and $g_j$ is an extreme point of $T$. Define an affine map $\pi : S_u(G_0)\dot{\times}T \to T$ by $\pi((e_i, g_j)) = g_j$ if $g_j \in S_k$, and $\pi(e_i, g_j) = f_i$ if $g_j$ is not in any of $S_k$. Since there are only finitely many extreme points in both $S_u(G_0)$ and $T$, $\pi$ is a continuous affine surjective map.

Then

$$\lambda_T \circ \pi = \alpha.$$

Choose a lifting $\iota : T \to S_u(G_0)\dot{\times}T$ of $\pi$ by $\imath(g_j) = (\lambda_T(g_j), g_j)$ for $g_j \in S_j$, $j = 1, 2, ..., n$. In particular, $\pi \circ \iota = \mathrm{id}_T$. Define an affine map $\beta : S_u(G_0)\dot{\times}T \to S_u(G_0)\dot{\times}T$ by $\beta = \iota \circ \pi$.

By Theorem 5.18, there is a $\mathcal{Z}$-stable ASH-algebra $A' \in \mathcal{A}$ with

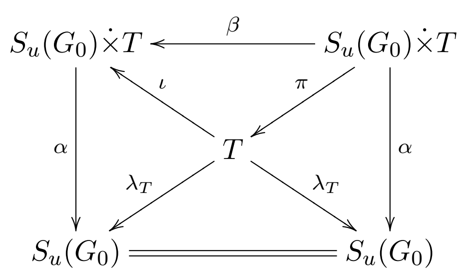
$$\mathrm{Ell}(A') = (G_0, G_1, S_u(G_0)\dot{\times}T, \alpha).$$

By Theorem 6.2, there is a unital homomorphism $\varphi : A' \to A'$ such that

$$[\varphi]_0 = \mathrm{id}, \quad [\varphi]_1 = \mathrm{id}, \quad \text{and} \quad (\varphi)^\sharp = \beta.$$

(The compatibility between the map $\beta$ and $[\varphi]_0$ follows from the commutative diagram below.) Let $A_n = A'$ and let $\varphi_n : A_n \to A_{n+1}$ be defined by $\varphi_n = \varphi$. $n = 1, 2, ....$ Put $A = \lim_{n\to\infty}(A_n, \varphi_n)$. Since each $A_n$ is simple so is $A$. By Theorem 11.10 of [18], $A \in \mathcal{A}$. Since each $A_n$ is an ASH-algebra, the $C^*$-algebra $A$ can be locally approximated by subhomogeneous $C^*$-algebras.

Since the diagram




commutes (the left triangle commutes because $\alpha \circ \iota = \lambda_T \circ \pi \circ \iota = \lambda_T \circ \mathrm{id}_T = \lambda_T$), one has that the inductive limit $A = \varinjlim(A', \varphi)$ satisfies

$$\mathrm{Ell}(A) = (G_0, G_1, T, \lambda_T),$$

as desired. □

**Lemma 6.6.** *Let $G$ be a countable rationally Riesz group and let $T$ be a metrizable Choquet simplex. Let $\lambda : T \to S_u(G)$ be a surjective affine map preserving extreme points. Then, there are decompositions $G = \varinjlim(G_n, \psi_n)$, $\mathrm{Aff}T = \varinjlim(\mathbb{R}^{k_n}, \eta_n)$, and maps $\lambda_n : G_n \to \mathbb{R}^{k_n}$ such that each $S_u(G_n)$ is a simplex with finitely many extreme points, and the following diagram commutes:*

$$\begin{array}{ccccccc}
\mathbb{R}^{k_n} & \xrightarrow{\eta_n} & \mathbb{R}^{k_{n+1}} & \longrightarrow & \cdots & \longrightarrow & \mathrm{Aff}T \\
\uparrow{\scriptstyle \lambda_n} & & \uparrow{\scriptstyle \lambda_{n+1}} & & & & \uparrow{\scriptstyle \lambda^*} \\
G_n & \xrightarrow[\psi_n]{} & G_{n+1} & \longrightarrow & \cdots & \longrightarrow & G
\end{array}.$$

*Proof.* Consider the ordered group $H = G \otimes \mathbb{Q}$. It is clear that the ordered group $G_{\mathrm{F}} := G/\mathrm{Tor}(G)$ (with positive cone the image of the positive cone of $G$) is a sub-ordered-group of $H$. Since $G$ is a rationally Riesz group, the group $H$ is a Riesz group. It follows from Effros-Handelman-Shen Theorem that there is a decomposition $H = \varinjlim(H_i, \varphi_i)$, where $H_i = \mathbb{Z}^{m_i}$

with the usual order for some natural number $m_i$. We may assume that the images of $(\varphi_{i,\infty})$ is increasing in $H$. Set

$$G_i' = \varphi_{i,\infty}^{-1}(G_{\mathrm{F}} \cap \varphi_{i,\infty}(H_i)) \subseteq H_i.$$

Then the inductive limit decomposition of $H$ induces an inductive limit decomposition

$$G_{\mathrm{F}} = \varinjlim(G_i', \varphi_i|_{G_i'}).$$

We assert that $S_u(G_i') = S_u(H_i)$. To show the assertion, it is enough to show that any state on $H_i$ is determined by its restriction to $G_i'$, and it is enough to show that two states of $H_i$ are same if their restrictions to $G_i'$ are same. Indeed, let $\tau_1$ and $\tau_2$ be two states of $H_i$ with same restrictions to $G_i'$. For any element $h \in H_i$, consider $\varphi_{i,\infty}(h) \in H$. We then can write

$$\varphi_{i,\infty}(h) = \frac{p}{q}g,$$

for some $g \in G_{\mathrm{F}}$ and relatively prime numbers $p$ and $q$. In particular,

$$pg = q\varphi_{i,\infty}(h) \in \varphi_{i,\infty}(H_i) \cap G_{\mathrm{F}},$$

and hence

$$qh \in G_i'.$$

Therefore, one has

$$\tau_1(h) = \frac{1}{q}\tau_1(qh) = \frac{1}{q}\tau_2(qh) = \tau_2(h).$$

This proves the assertion.

Since $S_u(H_i)$ is a finite dimensional simplex, the convex set $S_u(G_i')$ is also a simplex with finitely many extreme points. Hence we have the inductive decomposition $G_{\mathrm{F}} = \varinjlim(G_i', \psi_i')$, where $\psi_i' = \varphi_i|_{G_i'}$.

Consider the extension

$$0 \longrightarrow \mathrm{Tor}(G) \xrightarrow{\iota} G \xrightarrow{\pi} G_{\mathrm{F}} \longrightarrow 0,$$

and write $\mathrm{Tor}(G) = \varinjlim(T_i, \iota_i)$, where $T_i$ are finite abelian groups. Since the torsion free abelian group $G_1'$ is finitely generated, there is a lifting $\gamma_1 : G_1' \to G$ with $\pi \circ \gamma_1(g) = \psi_{i,\infty}'(g)$ for any $g \in G_1'$. Since an element in $G$ is positive if and only if it is positive in the quotient $G_{\mathrm{F}}$, it is clear that $\gamma_1$ is positive. Consider the ordered group $T_1 \oplus G_1'$ with the order determined by $G_1'$, and set the map $\psi_{1,\infty} : T_1 \oplus G_1' \to G$ by $(a, b) \mapsto \iota(a) + \gamma_1(b)$. It is clear that $\psi_{1,\infty}$ is positive.

Using the same argument, one has a positive lifting $\gamma_2 : G_2' \to G$. Since $\psi_{i,\infty}'(G_1') \subseteq \psi_{2,\infty}'(G_2')$, one has that $\gamma_2(g) - \gamma_1(g) \in \mathrm{Tor}(G)$ for each $g \in G_1'$. By truncating the sequence $(T_i)$, one may assume that $(\gamma_2 - \gamma_1)(G_1') \in T_2$. Define the map $\psi_{2,\infty} : T_2 \oplus G_2' \to G$ by $(a, b) \mapsto \iota(a) + \gamma_2(b)$. Then it is clear that $\psi_{1,\infty}(T_1 \oplus G_1') \subseteq \psi_{2,\infty}(T_2 \oplus G_2')$. Define the map $\psi_{1,2} : T_1 \oplus G_1' \to T_2 \oplus G_2'$ by

$$(a, b) \mapsto (\iota(a) + \gamma_1(b) - \gamma_2(b), \psi_{1,2}'(b)).$$

A direct calculation shows that $\psi_{1,\infty} = \psi_{2,\infty} \circ \psi_{1,2}$.

Repeating this argument and setting $G_i = T_i \oplus G_i$, one has the inductive limit decomposition

$$G = \varinjlim_i(G_i, \psi_{i,i+1}).$$

Noting that the order on $G_i$ is determined by the order on $G_i'$, one has that $S_u(G_i) = S_u(G_i')$, and hence the convex set $S_u(G_i)$ is a simplex with finitely many extreme points.

Let $\{a_n\}$ be a dense sequence in the positive cone $\mathrm{Aff}^+T$. Consider the map $\lambda^* \circ \psi_{i,\infty} : G \to \mathrm{Aff}T$. Since $S_u(G_i) = S_u(H_i)$ and $H_i = \mathbb{Z}^{m_i}$, the image of positive elements of $G_i$ is contained in a finite dimensional cone. Since images of $G_i$ are increasing, we may choose $\{b_1, ..., b_i, ...\} \subseteq \mathrm{Aff}T$ and natural numbers $n_1 < \cdots < n_i < \cdots$ such that $\{b_1, ..., b_{n_i}\}$ is a set of generators for the image of $G_i$ in $\mathrm{Aff}T$.

For each $i$, set $k_i = i + n_i$. We identify the affine space

$$\mathbb{R}^{k_i} \cong (\mathbb{R}a_1 \oplus \cdots \oplus \mathbb{R}a_i) \oplus (\mathbb{R}b_1 \oplus \cdots \oplus \mathbb{R}b_{n_i})$$

as the subspace of $\mathrm{Aff}T$ spanned by $a_1, ..., a_i, b_1, ..., b_{n_i}$. Define the map $\lambda_i : G_i \to \mathbb{R}^{k_i}$ by

$$g \mapsto (0 \oplus \cdots \oplus 0) \oplus (\lambda^* \circ \psi_{i,\infty}(g)),$$

the map $\eta_i : \mathbb{R}^{k_i} \to \mathbb{R}^{k_{i+1}}$ by

$$(f, g) \mapsto \iota_1(f) \oplus \iota_2(g),$$

where $\iota_1$ and $\iota_2$ are the inclusions of $\mathbb{R}a_1 \oplus \cdots \oplus \mathbb{R}a_i$ and $\mathbb{R}b_1 \oplus \cdots \oplus \mathbb{R}b_{n_i}$ to $\mathbb{R}a_1 \oplus \cdots \oplus \mathbb{R}a_{i+1}$ and $\mathbb{R}b_1 \oplus \cdots \oplus \mathbb{R}b_{n_{i+1}}$ in $\mathrm{Aff}T$, respectively. Then, it is a straightforward calculation that $\mathrm{Aff}T$ has the decomposition $\varinjlim(\mathbb{R}^{k_i}, \eta_i)$, and the diagram in the lemma commutes. □

Finally, we reach the main result of this paper.

**Theorem 6.7.** *Let $(G_0, (G_0)_+, u)$ be a countable partially ordered weakly unperforated and rationally Riesz group, let $G_1$ be a countable abelian group, let $T$ be a metrizable Choquet simplex and let $\lambda_T : T \to S_u(G_0)$ be a surjective affine continuous map sending extremal points to extremal points. Then there exists one (and exactly one, up to isomorphic) unital $\mathcal{Z}$-stable $C^*$-algebra $A \in \mathcal{A}$ such that*

$$\mathrm{Ell}(A) = ((G_0, (G_0)_+, u), G_1, T, \lambda_T).$$

*Moreover, $A$ can be constructed to be locally approximated by subhomogeneous $C^*$-algebras.*

*Proof.* Note that the part of the statement about "exactly one, up to isomorphism" follows from [18].

By Lemma 6.6, there exists a decomposition

$$\begin{array}{ccccccc}
\mathbb{R}^{k_n} & \xrightarrow{\eta_n} & \mathbb{R}^{k_{n+1}} & \longrightarrow & \cdots & \longrightarrow & \mathrm{Aff}T \\
\uparrow{\scriptstyle \lambda_n} & & \uparrow{\scriptstyle \lambda_{n+1}} & & & & \uparrow{\scriptstyle \lambda_T^*} \\
K_n & \xrightarrow[\psi_n]{} & K_{n+1} & \longrightarrow & \cdots & \longrightarrow & G_0
\end{array},$$

where each $K_n$ is a rationally Riesz group with $S_u(K_n)$ having finitely many extreme points. By Lemma 6.5, there is a unital $\mathcal{Z}$-stable algebra $A_n \in \mathcal{A}$ such that

$$((K_0(A_n), K_0(A_n)_+, [1_{A_n}]), K_1(A_n), \mathrm{Aff}(\mathrm{T}(A_n)), \lambda_{A_n}) = ((K_n, (K_n)_+, u), G_1, \mathbb{R}^{k_n}, \lambda_n),$$

and each $A_n$ can be locally approximated by subhomogeneous C*-algebras.

By Theorem 6.2, there are *-homomorphisms $\varphi_n : A_n \to A_{n+1}$ such that $(\varphi_n)_{*0} = \psi_n$, $(\varphi_n)_{*1} = \mathrm{id}_{G_1}$ and $(\psi_n)_* = \eta_n$, where $(\psi_n)_*$ is the induced map from $\mathrm{Aff}(\mathrm{T}(A_n)) \to \mathrm{Aff}(\mathrm{T}(A_{n+1}))$. Then the inductive limit $A = \varinjlim_n(A_n, \psi_n)$ is in the class $\mathcal{A}$ and satisfies

$$\mathrm{Ell}(A) = ((G_0, (G_0)_+, u), G_1, T, \lambda_T).$$

Since each $A_n$ can be locally approximated by subhomogeneous C*-algebras, so does the $C^*$-algebra $A$. □

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