A CLASSIFICATION FOR TRACIALLY SPLITTING TREE ALGEBRAS

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Abstract. A class of unital C*-algebras (TAS-algebras for short) which can be tracially approximated by splitting interval algebras (a certain class of sub-C*-algebras of interval algebras) is introduced. It is shown that the class of simple separable nuclear TAS-algebras which satisfy the UCT is classified by the Elliott invariant.

1. Introduction

Elliott's program on the classification of nuclear C*-algebras is expanding rapidly in recent years, and covers abundance classes of C*-algebras. Among them, one important class is the class of unital simple inductive limits of homogeneous C*-algebras without dimension growth. Let us refer the C*-algebras in this class as AH-algebras in this paper. (In other words, all AH-algebras in this paper are assumed to have no dimension growth, which is a crucial condition, as plenty of exotic simple unital inductive limits of homogeneous C*-algebras have been constructed. See, for example, [28], [29], [27], etc.) This class of C*-algebras was classified in the works of G. A. Elliott, G. Gong, and L. Li (see [3], [7], and [4]).

The axiomization of AH-algebra is then given by H. Lin (see [13], [11], [12], [15], [18]). In stead of looking at C*-algebras of inductive limit, Lin considered tracially approximately interval algebras (TAI-algebra for short), the C*-algebras which can be approximated by interval algebras in the sense of Egolov. More precisely, a simple C*-algebra A is a TAI-algebra if for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $0 \neq a \in A^+$, there is a nonzero projection $p \in A$ and a C*-subalgebra $I \subset A$ with $1_I = p$ and $I \cong \mathrm{C}([0,1]) \otimes F$ for some finite dimensional C*-algebra F such that for all $x \in \mathcal{F}$,

(1) $\|xp - px\| < \varepsilon$,
(2) $pxp \in_{\varepsilon} I$, and
(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

Lin showed that the class of nuclear TAI algebras satisfying the Universal Coefficient Theorem is classifiable. Since AH-algebras are TAI-algebras ([7]) and they exhaust all the possible invariant, this class of C*-algebras coincides with the class of AH-algebras, and hence provides an axiomization of AH-algebras.

The axiomization of AH-algebras plays an important role in recent progresses of the classification program. First, the condition being tracially approximated by interval algebras is much easier to verified than the condition being an inductive limit of certain building blocks. Hence, it provides a powerful tool to verify whether some naturally arising C*-algebras fall inside the

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class of AH-algebras, see, for example, [21]. (Note that most of the earlier known results were restricted to AH-algebras with building blocks of one dimensional spectrum, which have stable relations.)

More important, Lin's axiomization of AH-algebras is also one of the key ingredients of the further development of the classification beyond AH-algebras. In recent works [16], [19], [30], [17], and [20], the classification theorem was pushed to the class of $\mathcal{Z}$-stable C*-algebras which are TAI-algebras after tensoring with the UHF-algebra Q. The K_0-groups of this class of C*-algebras were shown to exhaust all the ordered groups which has the Riesz decomposition property after tensoring with $\mathbb{Q}$ (see, [**?**]), the so-callled rationally Riesz groups.

In this paper, the method of Lin's axiomatization of AH-algebras is generalized to the classification of simple inductive limits of certain subhomogeneous C*-algebras. One considers the C*-algebras which can be tracially approximated by splitting tree algebras, which were introduced by H. Su in [26] (a particular subclass, splitting interval algebras, was studied further by X. Jiang and Su in [9]). It is shown that this class of C*-algebras contains all simple inductive limits of splitting tree algebras together with homogeneous C*-algebras without dimension growth.

Using the same strategy of Lin in [15] and [18], one obtains a classification theorem for the subclass consisting of those nuclear C*-algebras which can be tracially approximated by splitting interval algebras and satisfies the UCT. Even the K_0-groups of this class of C*-algebras do not exhaust all weakly unperforated ordered groups, it was known that the K_0-groups of C*-algebras in this subclass may fail to satisfy the Riesz decomposition property, even after tensoring with $\mathbb{Q}$ (see [**?**] and [9]). Hence this class of C*-algebras contains more C*-algebras than the class of TAI-algebras.

2. Splitting tree algebras and tracially approximately splitting tree algebras

In this section, after reviewing on the basic building blocks—splitting tree algebras, the tracially approximately splitting tree algebras will be introduced, and certain basic properties will be studied.

2.1. **Splitting tree algebras and their K-theory.** Let T be a finite path-connected tree. Denote its vertices by $\{v_i\}_{i=1,...,n}$, and its edges by $\{e_i\}_{i=1,...,m}$. We define a *splitting tree algebra* to be the algebra of matrix-valued functions over T which take the prescribed diagonal blocks form on the vertices. That is, for an integer $k > 0$ and $\overline{k}_1, ..., \overline{k}_n$, n partitions of $\{1, 2, ..., k\}$, we define the *splitting tree algebra* $S(\overline{k}_1, ..., \overline{k}_n; T)$ to be

$$\{f \in \mathrm{M}_k(\mathrm{C}(T)); f(v_i) \in \bigoplus_{j\in\overline{k}_i} \mathrm{M}_j(\mathbb{C}), i = 1, ..., n\}.$$

They are unital C*-subalgebras of homogeneous C*-algebras with one-dimensional spectrum. The spectrum of a splitting tree algebra is in fact a non-Hausdorff tree, which has the same edges as T, but it splits into several points according to the diagonal form at each vertex. We refer the vertices of T the *singular points* of the splitting tree algebra S. And we label the splitting points at v_i by $v_i^1, ..., v_i^{|\overline{k}_i|}$ respectively. This class of type I C*-algebras was introduced by Su in [26]. In the case that the tree T is the interval $[0, 1]$, the splitting tree algebras are referred as *splitting interval algebras*. In [9], the authors studied the unital simple inductive limits of splitting interval algebras, and showed that this class of C*-algebras is classified by the Elliott invariant.

Consider a splitting tree algebra S, and consider the ideal T consisting of matrix-valued functions vanishing at the vertices. We have the following exact sequence:

$$0 \to I \to S \to F \to 0,$$

where $F \cong \bigoplus_{i=1}^{n} \bigoplus_{i\in\overline{k}_i} \mathrm{M}_i(\mathbb{C})$. A calculation with the associated six-term exact sequence gives us the K-groups of S.

Lemma 2.1 ([26]). *The K_0-group of the splitting tree algebra $S(\overline{k}_1, \cdots, \overline{k}_n; T)$ is*

$$\left\{ (m_1, \cdots, m_n) \in \bigoplus_{i=1}^{n} \mathbb{Z}^{|\overline{k}_i|}; \sum_i m_1^{(i)} = \cdots = \sum_i m_n^{(i)} \right\}.$$

The ordered structure on the K_0-group inherits from the standard order on $\bigoplus_{i=1}^{n} \mathbb{Z}^{|\overline{k}_i|}$, and $[1] = (\overline{k}_1, \cdots, \overline{k}_n)$.

The K_1-group of $S(\overline{k}_1, \cdots, \overline{k}_n; T)$ is trivial.

There are several distinguished homomorphisms from the K_0-group of a splitting tree algebra to $\mathbb{Z}$: The standard quotient maps from $\bigoplus_{i=1}^{n} \mathbb{Z}^{|\overline{k}_i|}$ to each component. These homomorphisms are in fact induced by the point evaluation maps of the splitting tree algebras on the correspondence splitting points:

$$\mathrm{ev}_i^j : S \to \mathrm{M}_{\overline{k}_i(j)}(\mathbb{C}), \quad f \mapsto f(v_i^j).$$

Therefore, we also refer these distinguished K-theory maps *the distinguished point evaluation maps* (of the K_0-group of a given splitting tree algebra).

It is shown in [26] that the normalized traces over a splitting tree algebra are the probability measure on its spectrum. To be more precise, each normalized trace τ comes from a convex combination of a probability measure on the tree and normalized traces on the splitting points:

$$\tau(f) = \lambda_0 \int_0^1 \mathrm{tr}(f) d\mu + \sum_{i,j} \lambda_i^j \mathrm{tr}(f(v_i^j)), \quad \forall f \in S,$$

where tr is the normalized trace over a matrix algebra, μ is a probability measure concentrating on T, λ_0, $\lambda_i^j \geq 0$ and $\lambda_0 + \sum_{i,j} \lambda_i^j = 1$. Moreover, μ and λ's are uniquely determined if for each i, there is j such that $\lambda_i^j = 0$. Let us assume this condition in the sequel.

Extremal tracial states of S are induced by Dirac measures on the spectrum of S. They inherit the topology from the spectrum of S. That is, the extremal tracial states coming from regular points around a vertex v_i of T converge to the average of extremal tracial states over the splitting points on v_i according to the size of the diagonal blocks. Any tracial state induces a positive homomorphism (state) from the K_0-group to $\mathbb{R}$. An extremal tracial state of a splitting tree algebra may not induce an extremal state of its K_0-group (for example, the state induced by a

Dirac measure on a regular point is not an extremal state if there are more than on splitting points at some vertex).

On the other hand, any state of $\mathrm{K}_0(S)$ comes from a tracial state of S by the nuclearity of splitting interval algebras. Therefore, the distinguished point evaluations on the K_0-group span the convex of states on $\mathrm{K}_0(S)$. Moreover, if the image of a state of $\mathrm{K}_0(S)$ is isomorphic to $\mathbb{Z}$, we have the following lemma.

Lemma 2.2. *Let S be a splitting tree algebra. Then any positive homomorphism $\alpha : K_0(S) \to \mathbb{Z}$ is a sum of point evaluations.*

Proof. Let $S = S(\overline{k}_1, \cdots, \overline{k}_n; T)$. Then

$$\mathrm{K}_0(S) = \{(m_1, \cdots, m_n) \in \bigoplus_{i=1}^{n} (\mathbb{Z}^{|\overline{k}_i|}); \sum_j m_1^{(j)} = \cdots = \sum_j m_n^{(j)}\}.$$

Since α comes from a trace (may not be normalized), we have

$$\alpha((m_1, \cdots, m_n)) = \lambda_0 \sum_j m_1(j) + \sum_{i,j} \lambda_i^j m_i(j)$$

where λ's are positive numbers, and for any i, at least one of $\{\lambda_{i,j}\}$ is 0. Therefore, there exists a minimal projection p of S such that $[p]_0(i)(j) = 0$ whenever $\lambda_i^j \neq 0$. Apply α to this element; we get $\lambda_0 = \alpha([p]) \in \mathbb{Z}^+$.

In the same way, for any i_0, j_0 such that $\lambda_{i_0}^{j_0} \neq 0$, we can find a minimal projection p in S such that $[p]_0(i_0)(j_0) = 1$ and $[p]_0(i)(j) = 0$ whenever $\lambda_i^j \neq 0$ and $i \neq i_0$. Evaluating α on p, we get $\lambda_i^j = \alpha([p]) \in \mathbb{Z}^+$.

Therefore, all coefficients are positive integers, and hence α is a sum of the distinguished point evaluations on the K_0-group. □

Now, let us lift a positive homomorphism between the scaled ordered K_0-groups of two splitting tree algebras to a $*$-homomorphism between these C*-algebras. If the codomain algebra is a matrix algebra over the complex number, then the lemma above implies that the K-theory map is a sum of the point evaluations at splitting points. Since the K-theory map preserves the order unit $[1]_0$, the sum of the sizes of the matrix algebra equal to the size of the codomain algebra. Hence the direct sum of those point evaluation maps is the desired *-homomorphism.

In general setting, consider a homomorphism

$$\kappa : (\mathrm{K}_0(S_1), \mathrm{K}_0(S_1)^+, [1_{S_1}]_0) \to (\mathrm{K}_0(S_2), \mathrm{K}_0(S_2)^+, [1_{S_2}]_0)$$

for two splitting tree algebras S_1 and S_2. Denote the singular points of S_1 and S_2 by $\{w_i\}$ and $\{v_i\}$ respectively. Let us consider the composition of κ with a distinguished point evaluation map of $\mathrm{K}_0(S_2)$, say ev_i^j. Since the codomain of ev_i^j is $\mathbb{Z}$, $\mathrm{ev}_i^j \circ \kappa$ maps $\mathrm{K}_0(S_1)$ to $\mathbb{Z}$. By the argument above in the case the codomain is the matrix algebra, there is a homomorphism from S_1 to the matrix block at the splitting point coresponding to ev_i^j, and it is a sum of certain distinguished point evaluations of S_1.

Repeat this procedure for all splitting points of S_2. We shall show that for any two singular points (vertices of the tree) of S_2, the sets of the point evaluations corresponding to them can be matched to each other. Pick a singular point v_{i_0} of S_2, and denote the point evaluations of S_1 according to the point v to be $\{w_i^j\}$ with multiplicities $\{m_i^j\}$. Then we can group them in the following way: At each singular point w_i of S_1, set $m_i = \min_j\{m_i^j\}$. Then divide the point evaluations at this singular point into two groups: one group is the point evaluations at each splitting point with multiplicity m_i, the remaining point evaluations form the other group. In other words, we divide them into the point evaluations which are the same as a regular point with multiplicity m_i and the point evaluations which is not full at this singular point. After doing the same procedure at each singular point of S_1, we have that all the point evaluations according to v can be grouped into two parts: One part is the point evaluations which are the same as a regular point, we call it the *full* point evaluations part; the other part is the sum of point evaluations on some splitting points and it sends some minimal projection to 0, call it the *non-full* point evaluation part . Moreover, by the same argument in the proof of Lemma 2.2, this decomposition is unique on the K_0-group level.

After grouping the point evaluations for each singular point v_i of S_2, we can match them between v_i's in the following way. Note that for each singular point v_i of S_2, the map $\sum_j (\mathrm{ev}_i^j)_*$ is the same. So, by the uniqueness of this decomposition, the full point evaluations parts of each singular point of S_2 have the same multiplicity, and so do the non-full point evaluation parts. Therefore, the non-full part of point evaluations can be glued together easily by constant maps. For any two singular points v_i, v_j of S_2 which there is only one edge between them, we can connect the correspondence full (regular) point evaluations parts by any path between them in the tree of S_1. Hence we get the following proposition:

Proposition 2.3. *Let S_1, S_2 be two splitting tree algebras, and $\kappa : K_0(S_1) \to K_0(S_2)$ be a scaled ordered homomorphism. Then there exists a $*$-homomorphism $\varphi : S_1 \to S_2$ such that $\varphi_* = \kappa$.*

2.2. **Tracially approximately splitting tree algebras.**

Definition 2.4. A unital C*-algebra A is called a *tracially approximately splitting tree algebra* (TAS-algebra) if for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $0 \neq a \in A^+$, there is a projection $p \in A$ and a C*-subalgebra $S \subset A$ which is a splitting tree algebra with $1_S = p \neq 0$ such that for all $x \in \mathcal{F}$,

(1) $\|xp - px\| < \varepsilon$,
(2) $pxp \in_\varepsilon S$, and
(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

Any separable simple TAS-algebra are MF in sense of [1].

Lemma 2.5. *Any separable simple C*-algebra A which satisfies conditions (1) and (2) of Definition 2.4 can be embedded into the algebra $\prod_i M_{n_i}(\mathbb{C})/\bigoplus_i M_{n_i}(\mathbb{C})$ for some positive integers n_i.*

Proof. Let $\mathcal{F} = \{a_1, a_2, \cdots\}$ be a countable dense subset of the unit ball of A with $a_1 = 1$, and set $\varepsilon_n = 1/2^n$. Apply the tracial approximation property with $\mathcal{F}_n = \{a_1, \cdots, a_n\}$ and ε_n. Then there is a C*-subalgebra S_n and a projection p_n such that $p_n a_i p_n \in_\varepsilon S_n$ and $\|pa_i - a_i p\| \leq \varepsilon_n$

for all $a_i, 1 \leq i \leq n$. Take a point evaluation map π_n of S_n to M_{n_i}. Then π_n can be extended to a positive linear contraction from p_nAp_n to the matrix algebra. Denote it by π_n as above. Then $\Phi_n(x) = \pi_n(pxp)$ gives us a positive linear contraction from $\mathcal{A}$ to the matrix algebra with $\|\Phi_n(a_ia_j) - \Phi_n(a_i)\Phi_n(a_j)\| < \varepsilon_n$ and $\|\Phi_n(a_i{}^*) - \Phi_n(a_i)^*\| < \varepsilon_n$ for any $1 \leq i, j \leq n$.

Applying this procedure for each n, we get a sequence of positive linear contractions $\{\Phi_n\}$ with the approximation properties above. Therefore the map Φ from A to the asymptotic sequence algebra $\prod_i \mathrm{M}_{n_i}(\mathbb{C})/\bigoplus_i \mathrm{M}_{n_i}(\mathbb{C})$ induced by $(\Phi_1, \Phi_2, \cdots, \Phi_n, \cdots)$ is a $*$-homomorphism. Since $\Phi(1) = 1$, Φ is a non-zero map. By simplicity of A, the *-homomorphism Φ maps A ono-to-one into $\prod_i \mathrm{M}_{n_i}(\mathbb{C})/\bigoplus_i \mathrm{M}_{n_i}(\mathbb{C})$. □

In [6], separable simple TAS-algebra were shown to have the following properties.

Proposition 2.6 ([6])**.** *Any separable simple TAS-algebra has the following properties:*

(1) *has at least one tracial state, and thus is stably finite;*
(2) *has stable rank one;*
(3) *has the (SP) property;*
(4) *the strict order on projections are determined by traces, and thus has weakly unperforated K_0-group.*

Let G be a order-unit group. There is a canonical map $\rho : G \to \mathrm{Aff}(\mathrm{S}(G))$ defined by

$$g \mapsto (s \mapsto s(g)) \quad \text{for any } s \in \mathrm{S}(G).$$

For a simple TAS-algebra A, the following lemma shows that the image of $\mathrm{K}_0(A)$ is dense.

Lemma 2.7. *Let A be a simple TAS-algebra. The image of the canonical map $\rho : K_0(A) \to \mathrm{Aff}(S(K_0(A), K_0^+(A), [1]_0))$ is dense in $\mathrm{Aff}(S(K_0(A), K_0^+(A), [1]_0))$.*

Proof. Denote by E the linear subspace of $\mathrm{Aff}(\mathrm{S}(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1]_0))$ spanned by the image of $\mathrm{K}_0(A)$. Since E contains $\rho(1_A)$, the constant function 1, E contains all constant functions. Moreover, since $\mathrm{K}_0(A)$ separates the states, E separates the states. Thus the linear sunspace E is dense in $\mathrm{Aff}(\mathrm{S}(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1]_0))$ by Corollary 7.4 of [8].

To prove the lemma, we must show that $\rho(\mathrm{K}_0(A))$ is dense in E. It is enough to show that for any projection $p \in A$ and any positive real number $r < 1$, the positive affine function $r \cdot \rho([p])$ can be approximated by elements in $\rho(\mathrm{K}_0(A))$.

For any $\varepsilon > 0$, there exist $n, k \in \mathbb{N}$ such that

$$\frac{k-1}{n+1} \leq r \leq \frac{k+1}{n} \quad \text{and} \quad |\frac{k+1}{n} - \frac{k-1}{n-1}| \leq \varepsilon.$$

Since A is a TAS-algebra, one has that $[p] = [p'] + [p'']$ with $p'' \in S$ for some splitting tree algebra S, and $\tau([p'])$ is uniformly small (less than $\frac{1}{n}$) for all $\tau \in \mathrm{S}(\mathrm{K}_0(A))$. Furthermore, by the simplicity of A, we may assume that $p'' \in S$ has sufficiently large multiplicities at each splitting point of S such that p'' is almost divisible by n. In other words, we have that

$$p'' = p_1'' + p_2'' + \cdots + p_n'' + q'',$$

where $p_i'' \sim p_j''$ and $q'' \preceq p_1''$. Thus for any $\tau \in \mathrm{S}(\mathrm{K}_0(A))$,

$$\frac{k}{n+1}\tau([p'']) \leq \tau([p_1''] + \cdots + [p_k'']) \leq \frac{k}{n}\tau([p'']).$$

Since $\tau(p') \leq \frac{1}{n}$, it follows that

$$\frac{k-1}{n+1}\tau([p]) \leq \tau([p_1''] + \cdots + [p_k'']) \leq \frac{k+1}{n}\tau([p])$$

holds for every $\tau \in \mathrm{S}(\mathrm{K}_0(A))$, which implies

$$|r \cdot \rho([p]) - \rho([p_1''] + \cdots + [p_k''])| \leq \varepsilon.$$

Therefore, $\rho(\mathrm{K}_0(A))$ is dense in E, and hence it is dense in $\mathrm{Aff}(\mathrm{S}(\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1]_0))$ as desired. □

2.3. **TAS-algebras from inductive limits.** Let $\mathcal{SG}$ denote the class of finite direct sums of the following basic building blocks:

(1) splitting tree algebras, which provide torsion free part of K_0-group,
(2) continuous functions over $T_{2,k}$, which provide torsion part of K_0-group,
(3) continuous functions over $S^1 \vee \cdots \vee S^1 \vee T_{3,k}$, which provide K_1-group,

where $T_{2,k}$ is a 2-dimensional CW-simplex obtained by attaching a two-dimensional disk D to S^1 via a map $S^1(\cong \partial D) \to S^1$ of degree k, and $T_{3,k}$ is a three-dimensional CW simplex obtained by attaching a 3-dimensional ball B to S^2 via a map $S^2(\cong \partial B) \to S^2$ of degree k. The homogeneous C*-algebras in (2) and (3) are referred as the *Gong standard homogeneous C*-algebras* (see [7] and [4]). Note that C*-algebras in (1) has torsion free K_0-groups and trivial K_1-groups; the K_0-groups of C*-algebras in (2) has form $\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$, and K_1-groups are trivial; the K_0-groups of C*-algebras in (3) are isomorphic to $\mathbb{Z}$, and K_1 groups are isomorphic to $(\bigoplus \mathbb{Z}) \oplus (\mathbb{Z}/k\mathbb{Z})$.

By Theorem A of [6], for any TAS-algebra A, there is an simple unital inductive limit B of $\mathcal{SG}$ such that the Elliott invariant of A is isomorphic to the Elliott invariant of B. In fact, the following proposition shows that B is also a TAS-algebra.

Let X be a simplicial complex. Denote by $\mathcal{SH}$ the class of C*-algebra containing finite direct sum of splitting tree algebras and homogeneous C*-algebras $p\mathrm{M}_n(\mathrm{C}(X))p$. Then, we have the following proposition.

Proposition 2.8. *Let $A = \varinjlim(A_n, \psi_n)$ be a unital simple inductive limit of building blocks in $\mathcal{SH}$ such that the dimension of base spaces of A_n is uniformly bounded, then A is a TAS-algebra.*

Proof. We shall show that the maps between building blocks can be decomposed into the sum of two homomorphisms, such that the one of them factors through splitting tree algebras, and the image of the identity has small traces.

Let $\mathcal{F}$ be a finite subset of $A = \varinjlim A_n$. Let $\varepsilon > 0$, $a \in A^+$, and J be a positive integer such that $1/J < \inf\{\tau(a); \tau \in \mathrm{T}(A)\}$. By Proposition 2.6 of [6], the C*-algebra A has the (SP) property, thus we may assume that a is a projection. Moreover, we may assume that $\mathcal{F} \subset A_n = S_n \oplus H_n$ where S_n is a splitting tree algebra and H_n is a homogeneous C*-algebra.

Then for any $m > n$, the map $\psi_{n,m} : A_n \to A_m$ can be decomposed into $\psi'_{n,m} + \psi''_{n,m}$ such that either domain or codomain of $\psi'_{n,m}$ is a splitting tree algebra (then $\psi'_{n,m}$ factors through splitting tree algebras), and the map $\psi''_{n,m}$ is a map between homogeneous algebras.

Since A is simple, by Theorem 4.37 of [7], for sufficiently large m, the map $\psi''_{n,m}$ can be approximately (with respect to $\mathcal{F}$) decomposed into

$$\psi''_0 \oplus \psi''_1 \oplus \psi''_2$$

such that the map ψ''_1 factors through a finite dimensional C*-algebra and ψ''_2 factors through an interval algebras. Moreover, $[\psi''_0(1)] \leq J[\psi''_0(1)]$. Therefore, if we set $\psi^0_{n,m} = \psi''_0$ and $\psi^1_{n,m} = \psi'_{n,m} \oplus \psi''_1 \oplus \psi''_2$, we have that

$$||\psi_{n,m}(f) - \psi^0_{n,m}(f) \oplus \psi^1_{n,m}(f)|| < \varepsilon \quad \text{for any } f \in \mathcal{F}$$

and $\psi^1_{n,m}$ factors through splitting tree algebras. Moreover, since $[\psi''_0(1)] \leq J[\psi''_0(1)]$, we have that $\tau(\psi''_0(1)) < 1/J < \varepsilon < \tau(a)$ for any $\tau \in \mathrm{T}(A)$. An argument same to Theorem 4.9 of [6] shows that A has the property that the strict order on projections is determined by tracial states, thus we have that $\tau''_0(1)$ is Murray-von Neumann equivalent to a subprojection of a, and therefore, A is a TAS-algebra. □

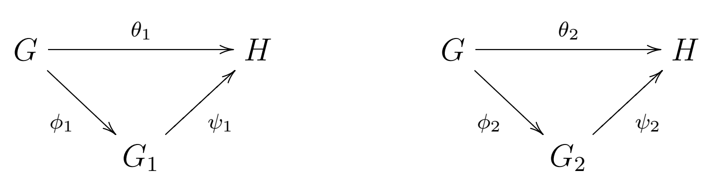
3. Existence Theorem

In this section, we shall lift homomorphisms between the Elliott invariant of TAS-algebras to approximate homomorphisms between the TAS-algebras (so called Existence Theorems).

3.1. **Existence theorem for maps from building block algebras to TAS-algebras.** Let κ be a homomorphism from the order-unit K_0-group of a splitting tree algebra S to the order-unit K_0-group of a TAS-algebra A. We show in this subsection that there exists a *-homomorphism $\phi : S \to A$ such that $[\phi]_0 = \kappa$.

Lemma 3.1 (Lemma 3.2 of [6]). *Let $G = K_0(A)$ where A is a splitting tree algebra, and let $r : G \to \mathbb{Z}$ be the point evaluation map on a regular point. Then there exist an $u \in G$ and a natural number m such that if the map $\theta : G \to G$ is defined by $g \mapsto r(g)u$, then the positive homomorphism $id + m\theta : G \to G$ factors through $\bigoplus_n \mathbb{Z}$ for some n.*

Lemma 3.2. *Let $G = K_0(A)$ where A is a splitting tree algebra and $H = K_0(B)$ for a simple TAS-algebra B. Then for any positive homomorphism $\theta : G \to H$, θ can be decomposed as $\theta_1 + \theta_2$ where θ_1 and θ_2 are positive homomorphisms from G to H such that the following diagrams commute:*



where $G_1 \cong \bigoplus_n \mathbb{Z}$ for some natural number n and G_2 is the K_0-group of a C-subalgebra of B which is a splitting tree algebra.*

Proof. Let m and u be as in Lemma 3.1. Since B has the (SP) property, there is an $h \in H^+$ such that h is less than the image of any generator of G^+. Define a positive homomorphism $\theta' : G \to H$ by $g \mapsto r(g)h$ where r is the point evaluation map on the regular point. Since h is sufficiently small, $\theta - \theta'$ is a positive homomorphism from G to H.

Since matrix algebras over TAS-algebras and any unital hereditary C*-subalgebra of a TAS-algebra is again a TAS-algebras (Lemma 2.3 of [6]), one can assume $\theta - \theta'$ is a positive homomorphism from G to $\mathrm{K}_0(B')$ where B' is a unital hereditary C*-subalgebra of B. Since B' is a TAS-algebra and the positive cone of G is finite generated, one has that $\theta - \theta'$ can be decomposed as the sum of two positive homomorphisms θ'_1 and θ_2, where θ_2 factors though the K_0-group of a C*-subalgebra of B'. Moreover, we may assume $m\theta'_1(u) < h$.

Therefore, we have $\theta = \theta' + \theta'_1 + \theta_2$. Since $m\theta'_1(u) < h$, we have a further decomposition of $\theta' + \theta'_1$: for any $g \in G$,

$$\begin{aligned}\theta'(g) + \theta'_1(g) &= r(g)h + \theta'_1(g) \\ &= r(g)(h - m\theta'_1(u)) + r(g)m\theta'_1(u) + \theta'(g) \\ &= r(g)(h - m\theta'_1(u)) + \theta'_1(mr(g)u) + \theta'_1(g) \\ &= r(g)(h - m\theta'_1(u)) + \theta'_1(mr(g)u + g).\end{aligned}$$

By Lemma 3.1, $g \to mr(g)u + g$ factors though $\bigoplus_n \mathbb{Z}$ for some n. Therefore, $\theta' + \theta'_1$ factors though $\bigoplus_{n+1} \mathbb{Z}$. Set θ_1 to be $\theta' + \theta'_1$, and it satisfy the lemma. □

With these lemmas, we have the following existence theorem for splitting tree algebras.

Theorem 3.3. *Let S be a splitting tree algebra, and let A be a simple TAS-algebra. Then, for any positive homomorphism $\kappa : (K_0(S), K_0^+(S), [1_S]_0) \to (K_0(A), K_0^+(A), [1_A]_0)$, there exists a $*$-homomorphism $\phi : S \to A$ such that $\phi_* = \kappa$.*

Proof. By Lemma 3.2, one has $\kappa = \kappa_1 + \kappa_2$ where κ_1 factors through $\bigoplus_n \mathbb{Z}$ for some natural number n, and κ_2 factors through the K_0-group of a splitting tree algebra which is a sub-C*-algebra of A. Then κ_2 can be lifted to a point evaluation map ϕ_2 from S to A by Proposition 2.3. Since κ_1 factors through $\bigoplus_n \mathbb{Z}$, there exists a $*$-homomorphism ϕ_1 from S to $A \otimes \mathcal{K}$ such that the induced K_0 map is κ_1. Therefore, the map $\phi_1 \oplus \phi_2 : S \to A \otimes \mathcal{K}$ induces $\kappa = \kappa_1 + \kappa_2$ on $\mathrm{K}_0(S)$. Let $p = (\phi_1 \oplus \phi_2)(1_S) \in A \otimes \mathcal{K}$. Then p is Murray-von Neumann equivalent to 1_A. Therefore, there is a partial isometry $v \in A \otimes \mathcal{K}$ such that $x \mapsto vxv^*$ is an *-isomorphism from $p(A \otimes \mathcal{K})p$ to A, and hence the map $\phi = v(\phi_1 \oplus \phi_2)v^*$ sends S to A. It is clear that induced K_0-map of ϕ is κ. □

The existence theorem for maps from a homogeneous C*-algebra to a TAS-algebra will be similar to that for TAI algebras [18]. First, we define a local existence property for KL-elements.

Definition 3.4. A C*-algebra A is said to be *KK-attainable* (for TAS-algebras) if for any simple TAS-algebra B, any $\alpha \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))^+$, and any finite subset $\mathcal{P} \subset P(A)$ with $[1_A] \in \mathcal{P}$, there is a sequence of completely positive linear contractions $L_n : A \to B \otimes \mathcal{K}$ such that for any $a, b \in A$,

$$\|L_n(ab) - L_n(a)L_n(b)\| \to 0,$$

and

$$[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$$

for sufficiently large n.

Remark 3.5. Note that for any splitting tree algebra A, it has finite generated torsion free K_0-groups and trivial K_1-groups, and satisfies the Universal Coefficient Theorem. Then one has that $\mathrm{KL(A, B)} \cong \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B))$ for any C*-algebra B. Hence, by Theorem 3.3, splitting tree algebras are always KK-attainable.

Let us consider the KK-attainability for homogeneous C*-algebras. First, we have the following Lemma.

Lemma 3.6 (Lemma 9.8 of [18]). *Let A be a unital C*-algebra, and let B be a unital separable simple TAS-algebra and S be a sub-C*-algebra of B which is a splitting interval algebra. Let G be a subgroup generated by $\mathrm{P}(A)$. If there is an $\mathcal{F}-\delta$-multiplicative contractive completely positive linear map $\psi : A \to S \subset B$ such that $[\psi]_G$ is well defined. Then, for any $\varepsilon > 0$, there exists a splitting interval sub-C*-algebra $C \subset B$ and an $\mathcal{F}-\delta$-multiplicative contractive completely positive linear map $\psi : A \to C \subset B$ such that*

$$[L]|_{G\cap K_0(A,\mathbb{Z}/k\mathbb{Z})} = [\psi]|_{G\cap K_0(A,\mathbb{Z}/k\mathbb{Z})},$$

and $\tau(1_C) < \varepsilon$ for any tracial state τ on B and any $k > 1$ with $G \cap K_0(A, \mathbb{Z}/k\mathbb{Z}) \neq \{0\}$, where L and ψ are viewed as maps to B. Furthermore, if $[L]|_{G\cap K_0(A,\mathbb{Z}/k\mathbb{Z})}$ is positive, so is $[\psi]|_{G\cap K_0(A,\mathbb{Z}/k\mathbb{Z})}$.

Proof. The proof is a modification of the proof of Lemma 9.8 of [18]. Without loss of generality, let us assume that

$$S = \{f \in \mathrm{C}([0, 1], \mathrm{M}_n);\ f(0) \in \bigoplus_i M_{m_i} \text{ and } f(1) \in \bigoplus_j M_{n_j}\}.$$

Let $\{q_k\}$ be a set of minimal projections of S which spans $\mathrm{K}_0(S)$. Suppose that

$$G \cap \mathrm{K}_0(A, \mathbb{Z}/k\mathbb{Z}) = \{0\} \quad k > K.$$

By Theorem 4.1 in Section 4, with $n = K! + 1$ and ε, we have that there are homomorphism $S \to C \oplus \underbrace{S_0 \oplus \cdots \oplus S_0}_{n}$, $x \mapsto L_0(x) \oplus L_1(x) \oplus \cdots \oplus L_1(x)$ with each unit of S_0 is Murray-von Neumann equivalent, and $\tau(1_C) \leq \varepsilon$, where C and S_0 are splitting interval sub-C*-algebra of B. Denote by ϕ the map $S \to S_0 \oplus S_1$ by $x \mapsto L_0(x) \oplus L_1(x)$. One then has that $\phi(f) + K![L_1](f) = [f]$ for any $f \in \mathrm{K}_0(S)$. Define the map $L = \phi \circ \psi$, and denote by $j_1 : S \to B$ and $j_2 : C \to B$ be the embedding. Then, one has that $(j_1)_* = (j_2 \circ \phi)_*$ on $\mathrm{K}_0(S, \mathbb{Z}/k\mathbb{Z})$ for $k \leq K$. Since $\mathrm{K}_1(S) = \mathrm{K}_1(C) = 0$, both $[L]$ and $[\psi]$ map $\mathrm{K}_0(A, \mathbb{Z}/k\mathbb{Z})$ to $\mathrm{K}_0(B)/k\mathrm{K}_0(B)$ and factor though $\mathrm{K}_0(S, \mathbb{Z}/k\mathbb{Z})$. Therefore

$$[L]|_{G\cap \mathrm{K}_0(A,\mathbb{Z}/k\mathbb{Z})} = [\psi]|_{G\cap \mathrm{K}_0(A,\mathbb{Z}/k\mathbb{Z})}, \quad k = 1, ..., K,$$

as desired. □

Lemma 3.7 (Lemma 9.9 of [18]). *Let $C = M_n(C(X))$, where X is a path connected compact metric space with $K_0(C(X)) = \mathbb{Z} \oplus tor(K_0(C(X)))$, and $K_1(C(X))$ is finite generated, and $K_0^+(C) \subset \{(z, x) : z \in \mathbb{N}, x \in \mathbb{Z}, or, (z, x) = (0, 0)\}$. Then C is KK-attainable.*

Proof. The proof is the same as that of Lemma 9.9. Instead of using Lemma 9.8 of [18], one use Lemma 3.6, and instead considering finite C*-algebra F_1 (in the proof of Lemma 9.9 of [18]), one considers the splitting interval algebra S of Lemma 3.6, and the same argument works (only use the fact $\mathrm{K}_1(S, \mathbb{Z}/k\mathbb{Z}) = 0$). □

For the two-sphere S^2, since $\mathrm{K}_0(\mathrm{C}(S^2)) = \mathbb{Z} \oplus \mathbb{Z}$, it does not satisfy the lemma above. However, we can embed $\mathrm{C}(S^2)$ to an AF-algebra which is a sub-C*-algebra of the TAS-algebra with a suitable K_0-embedding.

Definition 3.8. An ordered group G is *weakly divisible* if for any $g \in G^+$, there exist $g_1, g_2 \in G^+$ such that $g = 2g_1 + 3g_2$, or equivalently, G is weakly divisible if for any $g \in G^+$ and any $M > 0$, there exist $g_1, \cdots, g_n \in G^+$ and integers $m_1, \cdots, m_n > M$ such that $g = m_1 g_1 + \cdots + m_n g_n$.

Lemma 3.9. *Let S^2 be the two dimensional sphere, and let A be a C*-algebras with weakly divisible K_0-group. Suppose that A has the cancellation of projection and the strict order on the projections is determined by the traces. Then for any order-unit group homomorphism $\kappa : K_0(S^2) \to K_0(A)$, there is a *-homomorphism $\gamma : C(S^2) \to A$ such that $\gamma_0 = \kappa$.*

Proof. The K_0-group of $\mathrm{C}(S^2)$ is $\mathbb{Z} \oplus \mathbb{Z}$ with the order determined by the first cordinator. Denote the positive elements $(1, 0)$ and $(1, 1)$ in $\mathrm{K}_0(\mathrm{C}(S^2))$ by p' and q' respectively. Set the image of p' under κ to be p.

Let $\{M_i\}$ be an increasing sequence of natural number tending to $+\infty$. Since $\mathrm{K}_0(A)$ is weakly divisible, there exist $p_1^{(1)}, p_1^{(2)}, \cdots, p_1^{(d_1)}$ in $\mathrm{K}_0(A)^+$ and $m_1^{(1)}, m_1^{(2)}, \cdots, m_1^{(d_1)}$ such that $m_1^{(k)} > M_1$ for each k, and

$$\sum_{k=1}^{d_1} m_1^{(k)} p_1^{(k)} = p.$$

For each $p_1^{(k)}$, since $\mathrm{K}_0(A)$ is weakly divisible, there are positive elements $p_2^{(k,1)}, p_2^{(k,2)}, \cdots, p_2^{(k,d_{2,k})}$ in $\mathrm{K}_0(A)$ and natural numbers $m_2^{(k,1)}, m_2^{(k,2)}, \cdots, m_2^{(k,d_{2,k})}$ with $m_2^{(k,i)} > M_2$ for all $1 \le i \le n_{2,k}$ such that

$$\sum_{i=1}^{d_{2,k}} m_2^{(k,i)} p_2^{(k,i)} = p_1^{(k)}.$$

One can repeat this procedure for each positive elements $p_n^{(k_1,\cdots,k_n)}$ and get its further decomposition with the multiplicities larger that M_n. With these positive element, one can construct a dimension group embedding inside $\mathrm{K}_0(A)$ as following:

Set $D_1 = 1$ and set $D_n = \sum_{i=1}^{D_{n-1}} d_{n-1,i}$ for $n \ge 2$. In other words, D_n is the number of the positive elements $p_n^{(k_1,\cdots,k_n)}$ in n^{th} step. One can set

$$G'_n = \bigoplus_{D_n} \mathbb{Z},$$

and set the map ϕ_n from G'_n to G'_{n+1} by

$$\phi_n : e_n^{(k_1,\cdots,k_n)} \mapsto \sum_{j=1}^{d_{n,k}} m_{n+1}^{(k_1,\cdots,k_n,j)} e_{n+1}^{(k_1,\cdots,k_n,j)}$$

where $\{e_n^{(k_1,\cdots k_n)}\}$ is the standard basis of G'_n and $\{e_{n+1}^{(k_1,\cdots k_n,j)}\}$ are standard basis for G'_{n+1}. Denote the inductive limit of (G'_n, ϕ_n) by G'.

Note that there are positive maps $\iota_n : G'_n \to \mathrm{K}_0(A)$ by

$$\iota_n : e_n^{(k_1,\cdots k_n)} \to p_n^{(k_1,\cdots k)}.$$

It is easy to verify that $\{\iota_n\}$ are compatible with the maps $\{\phi_n\}$. Thus, there is a positive homomorphism $\iota : G' \to \mathrm{K}_0(A)$. Let $u \in G'$ be $\phi_{1,\infty}(1)$. It follows that

$$\iota(u) = p.$$

Define an ordered group

$$G = G' \oplus \mathbb{Z}$$

with the order determined by the first coordinator (in other words, $(g, n) > 0$ in G if and only if $g > 0$ in G'). One has a group homomorphism by $\theta : G \to \mathrm{K}_0(A)$ by

$$\theta : (g, n) \mapsto \iota(g) + n(q - p).$$

Since the order on the projections in A is determined by the traces and A is simple, one concludes that θ is a positive homomorphism. Note that $\theta(u, 0) = p$ and $\theta(u, 1) = p$.

On the other hand, one has an positive homomorphism $\psi : \mathrm{K}_0(\mathrm{C}(S^2)) \to G$ by

$$p' \mapsto (u, 0)$$

$$q' \mapsto (u, 1).$$

A simple calculation shows that

$$\kappa = \theta \circ \psi.$$

Since the multiplicities of building blocks G' is unbounded ($M_i \to +\infty$), there is no minimal elements in G'^+. Hence G satisfies the Riesz property, which implies G is a dimension group without minimal elements. Thus, there is an unital AF algebra B with

$$(\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B]_0) \cong (G, G^+, (u, 0)).$$

Since A has the cancellation of projections, there is a *-homomorphism $h : B \to A$ such that $h_0 = \theta$.

Consider the positive homomorphism $\psi : \mathrm{K}_0(\mathrm{C}(S^2)) \to \mathrm{K}_0(B)$. Since there is no minimal positive elements in G, there is a homomorphism $r : \mathrm{C}(S^2) \to \mathrm{K}_0(B)$ with $r_0 = \psi$ by [5]. Therefore, the *-homomorphism

$$\gamma = h \circ r$$

satisfies the theorem. □

Theorem 3.10. *Let A be a simple TAS-algebra. Then the group $G = K_0(A)$ is weakly divisible.*

Proof. By Corollary 3.4 of [6], K_0-group of a simple TAS-algebra A is a simple inductive limit of K_0-groups of splitting tree algebras and ordered groups $\mathbb{Z}\oplus\mathbb{Z}/n\mathbb{Z}, n=1,2,....$ Therefore, we may assume that $G=\varinjlim(G_i,\phi_i)$, where G_i is a direct sum of K_0-groups of splitting interval algebras and ordered groups $\mathbb{Z}\oplus\mathbb{Z}/n\mathbb{Z}, n=1,2,....$

Fix $N>10$. For any $a\in G^+$, since G is simple, we may assume that $a\in G_i$ with the multiplicity of a in each direct summand of G_i is larger than N (when we say a positive element in the K_0-group of a splitting tree algebra has multiplicity larger than N if it has multiplicity larger than N at each splitting point).

It is clear that the restriction of a to each group $\mathbb{Z}\oplus\mathbb{Z}/n\mathbb{Z}, n=1,2,...$ can be decompose as $2a_1+3a_2$ for some positive element a_1 and a_2.

Consider the restriction of a to a direct summand of G_i which is in the K_0-group of a splitting tree algebra. Then, it can be identified with the group

$$\left\{((m_1^{(1)},...,m_1^{(l_1)}),...,(m_n^{(1)},...,m_n^{(l_n)}))\in\bigoplus_{i=1}^{n}\mathbb{Z}^{l_i};\sum_i m_1^{(i)}=\cdots=\sum_i m_n^{(i)}\right\}$$

for some $m_j^{(i)}$, and identify the restriction of a by

$$((a_1^{(1)},...,a_1^{(l_1)}),...,(a_n^{(1)},...,a_n^{(l_n)}))$$

with $a_j^{(i)}>N$. Denote by $S=\sum_i m_1^{(i)}$. Consider the first singular point ($j=1$). There are positive integers $c_1^{(i)}$ and $d_1^{(i)}$ such that

$$a_1^{(i)}=2c_1^{(i)}+3d_1^{(i)}.$$

Denoted by $S_1=\sum c_1^{(i)}$ and $S_2=\sum d_1^{(i)}$. One can choose $c_1^{(i)}$'s and $d_1^{(i)}$ such that $|2S_1-S/2|\le 6$ and $|3S_2-S/2|\le 6$.

Consider the second singular point ($j=2$). Let us choose positive integers $c_2^{(i)}$ and $d_2^{(i)}$ such that

$$a_2^{(i)}=2c_2^{(i)}+3d_2^{(i)},\quad 1\le j\le l_2-1,$$

and

$$\left|\sum_{i=1}^{k}2c_2^{(i)}-\sum_{i=1}^{k}3d_2^{(i)}\right|\le 5,\quad 1\le k\le l_2-1,$$

as following: For $k=1$, it is clear that there are $c_2^{(1)}$ and $d_2^{(1)}$ such that $a_2^{(1)}=2c_2^{(1)}+3d_2^{(1)}$ with $\left|2c_2^{(1)}-3d_2^{(1)}\right|\le 5$. Assume that $c_2^{(i)}$ and $d_2^{(i)}$ are chosen for $1\le i\le k$. Denote by

$$e=\sum_{i=1}^{k}2c_2^{(i)}-\sum_{i=1}^{k}3d_2^{(i)}.$$

Then $|e|\le 10$. Write

$$a_2^{(k+1)}=2c_2^{(k+1)}+3d_2^{(k+1)}$$

with

$$6\le 3d_2^{(k+1)}-2c_2^{(k+1)}\le 11.$$

We then have that

$$\left|\sum_{i=1}^{k+1} 2c_2^{(i)} - \sum_{i=1}^{k+1} 3d_2^{(i)}\right| \leq 5.$$

Thus, such $\{c_2^{(i)}\}$ and $\{d_2^{(i)}\}$ exist. In particular

$$\left|\sum_{i=1}^{l_2-1} 2c_2^{(i)} - \sum_{i=1}^{l_2-1} a_2^{(i)}/2\right| \leq 3,$$

and

$$\left|\sum_{i=1}^{l_2-1} 3d_2^{(i)} - \sum_{i=1}^{l_2-1} a_2^{(i)}/2\right| \leq 3.$$

Then one has that

$$\sum_{i=1}^{l_2-1} 2c_2^{(i)} \leq \sum_{i=1}^{l_2-1} a_2^{(i)}/2 + 10 \leq S/2 - a_2^{(l_k)} + 10 \leq 2S_1 - a_2^{(l_k)} + 10,$$

and hence

$$S_1 - \sum_{i=1}^{l_2-1} c_2^{(i)} \geq a_2^{(l_k)}/2 - 10 > 0.$$

Same argument shows that

$$S_2 - \sum_{i=1}^{l_2-1} b_2^{(i)} > 0.$$

Define

$$c_2^{(l_k)} = S_1 - \sum_{i=1}^{l_2-1} c_2^{(i)}$$

and

$$d_2^{(l_k)} = S_2 - \sum_{i=1}^{l_2-1} d_2^{(i)}.$$

One then has that $\sum_{i=1}^{l_1} c_1^{(i)} = S_1 = \sum_{i=1}^{l_2} c_2^{(i)}$ and $\sum_{i=1}^{l_1} d_1^{(i)} = S_2 = \sum_{i=1}^{l_2} d_2^{(i)}$.

The same argument shows that for any k, there are $\{c_k^{(1)}, ..., c_k^{(l_k)}\}$ and $\{d_k^{(1)}, ..., d_k^{(l_k)}\}$ such that $S_1 = \sum_{i=1}^{l_k} c_k^{(i)}$ and $S_2 = \sum_{i=1}^{l_k} d_k^{(i)}$, and $a_k^{(i)} = 2c_k^{(i)} + 3d_k^{(i)}$. Denoted by

$$c = ((c_1^{(1)}, ..., c_1^{(l_1)}), ..., (c_n^{(1)}, ..., c_n^{(l_n)})),$$

and

$$d = ((d_1^{(1)}, ..., d_1^{(l_1)}), ..., (d_n^{(1)}, ..., d_n^{(l_n)})).$$

Then one has that $c \in G_i$ and $d \in G_i$, and $a = 2c + 3d$, as desired. □

As a corollary, one has

Corollary 3.11. *The C^*-algebra $C(S^2)$ is KK-attainable for simple TAS-algebras.*

Remark 3.12. By definition of KK-attainability, we see that it is a local property. In other words, a locally KK-attainable C*-algebra is KK-attainable. In particular, inductive limits of KK-attainable C*-algebras are KK-attainable. If we consider the homogeneous C*-algebras which appear in Section 2.3, they satisfy Lemma 3.7, and thus inductive limits of them together with splitting tree algebras are KK-attainable. Therefore, for any simple TAS-algebra A, by Theorem A of [6], there is a KK-attainable C*-algebra B such that the Elliott invariant of B is isomorphic to the Elliott invariant of A.

3.2. **Local existence theorem for maps from TAS-algebras to inductive limit C*-algebras.** In this section, we will study the existence theorem from a TAS-algebra to an inductive limit C*-algebras of splitting tree algebras together with the homogeneous C*-algebras described as in Section 2.3. For any simple TAS-algebra A, a simple inductive limite C*-algebra B and a positive KL-element $\alpha \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))^+$, we are going to construct an approximate homomorphism which almost induces α. The procedure is a modification of Huaxin Lin's proof of the local existence theorem of TAF-algebras [15].

First, let us sketch the proof. For such C*-algebras A, B, and positive KL-element α, it follows that α can be realized as the "difference" of two approximate homomorphisms from A to $B \otimes \mathcal{K}$, and one of them has finite-dimensional range. With these two maps, we may construct an approximate homomorphism Ψ from A to $B \otimes \mathcal{K}$ which agrees with α on $\mathrm{K}_1(A)$ and the K-group with the coefficients of A. In order to get the desired map on $\mathrm{K}_0(A)$, we compress Ψ to a small corner of $B \otimes \mathcal{K}$, then construct a map from A to B which factors though a splitting tree algebra (a C*-subalgebra of A). This map will induce a suitable map between the K_0-groups. To find such a map, we need some divisibility conditions on $\mathrm{K}_0(B)$. Since this map factors through the "large piece" of A, we also need to take care the elements of $\mathrm{K}_0(A)$ that vanish on the traces of A.

Recall that a C*-algebra A is called RFD if there is a family of finite dimensional representations of A which separate points. Let A be a simple separable TAS-algebra. By 2.5, there is an increasing family of RFD sub-C*-algebras $\{A_n\}$ such that their union is dense in A. Let $\{x_1, x_2, \cdots, x_n, \cdots\}$ be a dense sequence of elements in the unit ball of A. For any finite subset $\mathcal{F}_1 \subset A_1$, $\delta_0 > 0$, and a homomorphism h_0 from A_1 to a finite dimensional C*-algebra F_0 which is non-zero on $\mathcal{F}_1$, we get a non-zero homomorphism $h' : F_0 \to A$ by Lemma 2.1 of [22] such that

(1) $\|ea - ae\| < \delta$ and
(2) $\|h' \circ h_0(a) - eae\| < \delta$

for all $a \in \mathcal{F}_1$, with $e = h'(1)$. Set $H = h' \circ h_0$. Since any unital hereditary subalgebra of a TAS-algebra is a TAS-algebra, and any splitting tree algebra is finitely presented and has stable relations [2], one can apply the argument of 2.4 of [15], and obtains a sequence of finite subsets $\mathcal{F}_0, \mathcal{F}_1, \cdots$ in the unit ball of A with dense union, a sequence of decreasing positive numbers $\{\delta_n\}$, a sequence of projections $\{q_n\} \subset A$, a sequence of spitting tree C*-subalgebras S_n with $1_{S_n} = q_n$ and a sequence of homomorphisms $h_{n+1} : S_n \to S_{n+1}$ such that

(1) $\|q_n x - x q_n\| < \delta_n/4$ for all $x \in \mathcal{F}_n$;
(2) $\mathrm{dist}(q_n x_i q_n, S_n) < \delta_n/16, i = 1, \cdots, n$;

(3) $\tau(1-q_n) < 1/2^{n+1}$ for all tracial states τ on A;
(4) $G_n \subset \mathcal{F}_{n+1}$, where G_n generates S_n;
(5) $\|L_{n+1}(a) - h_{n+1}(a)\| < \delta/16^n$ for all $a \in \{L_n(\mathcal{F}_n)\} \cup \{S_n\}$, where $L_n : A \to S_n$ is a contractive positive linear map with $L_n|_{S_n} = \mathrm{id}_{S_n}$.

As in [15], let $S_{n,1}, S_{n,2}, \cdots, S_{n,m(n)}$ be the summands of S_n corresponding to each connected spectrum, and $\pi_{n,i} : S_n \to S_{n,i}$ be the quotient map. Let $\Psi_n : \mathcal{A} \to (1-q_n)A(1-q_n)$ be the map send a to $(1-q_n)a(1-q_n)$, and $J_n : A \to A$ be the map $a \mapsto \Psi_n(a) \oplus L_n(a)$. Note that Ψ_n and J_n are $\mathcal{F}_n - \delta_n/2$ multiplicative. Set $J_{m,n} = J_n \circ \cdots \circ J_m$ and $h_{m,n} = h_n \circ \cdots \circ h_m$. Note that $J_{m,n}$ is $\mathcal{F}_m - \delta_m$ multiplicative. We also use $L_n, \Psi_n, J_n, J_{m,n}, h_m$, and $h_{m,n}$ for their extensions on a matrix algebra of A. We have following lemma.

Lemma 3.13. *Let $\mathcal{P} \subset M_k(A)$ be a finite set of projections. Assume that $\mathcal{F}_n$ is sufficiently large and δ_0 is sufficiently small such that $[L_{n+1} \circ J_{1,n}]|_{\mathcal{P}}$ are well defined. Then*

$$\lim_{n\to\infty} \tau([L_{n+1} \circ J_{1,n}]([p])) = \tau([p])$$

for any $p \in \mathcal{P}, \tau \in T(A)$, with respect to uniform convergence on $T(A)$. Furthermore, for any projection q in S_1, we have

$$|\tau(h_{1+n} \circ \cdots \circ h_2(q)) - \tau(h_n \circ \cdots \circ h_2(q))| < 1/2^{n+1}$$

for all $\tau \in T(A)$, and

$$\lim_{n\to\infty} \tau(h_{1+n} \circ \cdots \circ h_2(q)) > 0$$

for all $\tau \in T(A)$.

The proof is the same as that of Lemma 2.7 of [15].

Remark 3.14. By Theorem 4.11 of [6], any positive state of the $\mathrm{K}_0(A)$ comes from a trace of A. The lemma above still holds if one replaces the trace τ by any positive state τ_0 on $\mathrm{K}_0(A)$.

As in [15], for a fixed finite subset $\mathcal{P}$ of the projections of A, and a natural number N, let $\tilde{\psi}_{N,i} = \pi_{N,i} \circ L_N$, $\tilde{\psi}_{N+1,i} = \pi_{N+1,i} \circ L_{N+1} \circ \Psi_N, \cdots, \tilde{\psi}_{N+n,i} = \pi_{N+n,i} \circ L_{N+n} \circ \Psi_{N+n-1}$. Then we have that

$$[L_{N+n} \circ J_{N,N+n+1}]_{\mathcal{P}} = \sum_{k,i} [\tilde{\psi}_{N+k,i}]_{\mathcal{P}}.$$

Therefore, if we rearrange $\{\tilde{\psi}_{N+m,i}\}$ as $\{\psi_i\}$, we have

$$\tau([L_{N+n} \circ J_{N,N+n+1}]([p]) = \sum_{j=1}^{s(n)} \tau([\psi_j]([p])).$$

The positive cone of the K_0-group of a splitting tree algebra is generated by the minimal projections. Therefore every projection can be written as the sum of the minimal projections $\{p_{i_1 \cdots i_k}\}$ ($p_{i_1 \cdots i_k}$ stands for the minimal projection which takes 1 at i_jth splitting point of the jth singular point). Moreover, we can make the expression to be unique if we only allow the minimal projections $\{p_{i \cdots i}\}$ which never cross between the splitting points, and the minimal projections $\{p_{1 \cdots 1 i \cdots i}\}$ which cross only once and starts at the first splitting point. Let

$\vec{p}_n = (p_{1\cdots 1}, \cdots, p_{k\cdots k}, p_{12\cdots 2}, \cdots, p_{1\cdots 1k})$ be the vector of these minimal projections. Then we have $\tau([\psi_j]([p])) = \vec{m}_n \cdot \tau([\vec{p}_n])^t$ where $\vec{m}_n$ is a vector of the multiplicities of p. Note that the entries of $\vec{p}_n$ satisfy the relations $[p_{1i}] + [p_{kk}] > [p_{1k}]$ at each singular point. Conversly, if there is a set of positive elements satisfying such relations in an ordered group, then there will ba a homomorphism from the K_0 group of the splitting tree algebra to the ordered group generated by them. Denote such vector of the minimal projections correspondence to ψ_i to be $\vec{p}_{i,n}$, then from Lemma 3.13 and Remark 3.14, we get the following lemma.

Lemma 3.15. *With notion same as above. For any $p \in \mathcal{P}$, we have that*

$$\tau([p]) = \lim_{n\to\infty} \sum_{i=1}^{s(n)} \vec{m}_i \cdot \tau(([\vec{p}_{i,n}])^t$$

uniformly on $S(K_0(A))$, and the affine maps induced by each entry of $\vec{p}_{i,n}$ converge to a positive map in $Aff(S(K_0(A)))$ as $n \to \infty$.

Remark 3.16. Moreover, we have a map $\tilde{\rho} : G \to \Pi\mathbb{Z}$ by

$$[p] \mapsto (\vec{m}_1(1), \cdots, \vec{m}_1(k_1), \vec{m}_2(1), \cdots, \vec{m}_2(k_2), \cdots),$$

where G is the sub-group of $K_0(A)$ generated by $\mathcal{P}$. By the lemma above, it is clear that if $\tilde{\rho}(g) = 0$, then $\tau(g) = 0$ for any trace over A.

By the definition of the map $\tilde{\rho}$ and H, the following lemma is clear.

Lemma 3.17. *Let $\mathcal{P}$ be a finite subset of projections in $M_k(A_1) \subset M_k(A)$. Then there is a finite subset $\mathcal{F}_1 \subset A_1$ and $\delta_0 > 0$ such that if the above construction starts with them, then*

$$\ker\tilde{\rho} \subset \ker[H] \quad \textit{and} \quad \ker\tilde{\rho} \subset \ker[h_0].$$

The K_0 part of the existence theorem will almost factor through the map $\tilde{\rho}$, and this lemma will help us to handle the elements of $K_0(A)$ which vanish under $\tilde{\rho}$. Moreover, to get a such K_0 homomorphism, we also need to find a copy of the generating set of the positive cone of $K_0(S)$ inside the codomain ordered group for some splitting tree algebra S. So, we will need the following lemma, which is a slight modification of a technical lemma in [15].

Lemma 3.18 ([15])**.** *Let S be a compact convex set and Aff(S) be the affine continuous functions on S. Let $\mathbb{D}$ be a dense ordered subgroup of Aff(S) and G be an ordered group with the strict order determined by a surjective homomorphism $\rho : G \to \mathbb{D}$. Let $\{x_{ij}\}_{0<i\leq r, 0<j<\infty}$ be an $r \times \infty$ matrix having rank r and with $x_{ij} \in \mathbb{Z}$ for each i, j and $g_j^{(n)} \in G$ such that $\rho(g_j^{(n)}) = a_j^{(n)}$, where $\{a_j^{(n)}\}$ is a sequence of positive elements in $\mathbb{D}$ such that $a_j^{(n)} \to a_j (>0)$ uniformly on S as $n \to \infty$. For each n, there is an $s(n)$ such that*

$$(x_{ij})_{r\times s(n)} \tilde{v_n} = \tilde{y_n}$$

where $\tilde{v_n} = (g_j^{(n)})_{s(n)\times 1}$ and $\tilde{y_n} = (\tilde{b}_j^{(n)}) \in G^r$. Set $b_i^{(n)} = \rho(\tilde{b}_i^{(n)})$ and $y_n = (b_j^{(n)})$. Suppose that $y_n \to z$ on S uniformly for some $z = (z_j)_{r\times 1}$.

Then there is a $\delta > 0$ and a positive integer $K > 0$ satisfying the following:

For some sufficiently large n, if $\tilde{z}' \in G^r$ *and there is* $\tilde{z}'' \in G^r$ *such that* $K^3\tilde{z}'' = \tilde{z}'$ *and* $\|z - Mz'\| < \delta$, *where* $z'_j = \rho(\tilde{z}'_j)$ *if* $\tilde{z}' = (\tilde{z}'_1, \cdots, \tilde{z}'_r)$, *and M is a positive integer, then there is an* $u = (\tilde{c}_j)_{s(n)\times 1} \in G_+^{s(n)}$ *such that*

$$(x_{ij})_{r\times sn} u = \tilde{z}'.$$

Moreover, we can choose $\{c_j^{(n)}\}_{j=1}^{s(n)}$ *such that for each n, if* $\{g_{j=1}^{(n)}\}_j^{s(n)}$ *generate the* K_0 *group of a splitting tree algebra as the minimal projections, so does* $\{c_{j=1}^{(n)}\}_j^{s(n)}$.

Proof. The proof is repeating the argument of Lemma 3.4 of [15]. We can verify that the u obtained in the [15] can be chosen to be sufficiently close to $\tilde{v}$ such that u also satisfies the relations $u_{1i}^{(k)} + u_{ij}^{(k)} > u_{1j}^{(k)}$ for each block correspondence to a singular point. Therefore, u generates a copy of K_0-group of the splitting tree algebra. □

Now, we are ready for the existence theorem from a TAS-algebra to a concrete algebras. First, we have the following theorem which holds in general setting. It lifts the given KL-element to the formal difference of two algebra maps.

Theorem 3.19 ([15]). *Let* $\mathcal{A}$ *be a separable C*-algebra satisfying UCT such that A is the closure of an increasing sequence* $\{A_n\}$ *of RFD C*-subalgebra and let B be a unital nuclear separable C*-algebra. Then for any* $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B))$, *there exist two sequences of completely positive contractions* $\phi_n^{(i)} : A \to B \otimes \mathcal{K}$ $(i = 1, 2)$ *satisfying the following:*

(1) $\|\phi_n^{(i)}(ab) - \phi_n^{(i)}(a)\phi_n^{(i)}(b)\| \to 0$ *as* $n \to \infty$,
(2) *for any n, the images of* $\phi_n^{(2)}$ *are contained in a finite dimensional C*-subalgebra of* $B\otimes\mathcal{K}$ *and for any finite subset* $\mathcal{P} \subset P(A)$, $[\phi_n^{(i)}]|_{\mathcal{P}}$ *are well defined for sufficiently large n,*
(3) *for each finite subset* $\mathcal{P} \subset P(A)$, *there exists* $m > 0$ *such that*

$$[\phi_n^{(1)}]|_{\mathcal{P}} = \alpha + [\phi_n^{(2)}]|_{\mathcal{P}}$$

for all $n > m$,
(4) *for each n, we may assume that* $\phi_n^{(2)}$ *is a homomorphism on* A_n.

By 2.5, any simple separable TAS-algebra satisfying UCT will satisfy the condition for $\mathcal{A}$ in the theorem. Therefore the theorem applies.

Recall that by an infinitesimal element of an ordered group (G, G^+), we mean an element of G which is vanished by all states of (G, G^+).

For any TAS-algebra A, by [6], there exists a simple inductive limit of splitting tree algebra and the Gong standard homogeneous C*-algebras such that it share with A the same Elliott invariant. In particular they have the same K-theory. Using this message, we can go a little further to show that there is a simple inductive limit of splitting tree algebra together with the Gong standard homogeneous C*-algebras and homogeneous C*-algebras over the two-sphere, such that it has the same K-theory with A, and there exists a map to a small corner of itself, which keeps all the invariants fixed except the K_0-group, and the difference on the K_0-group have any assigned multiplicity. More precise, we have:

Lemma 3.20. *Let A be a simple separable TAS-algebra. There exists a simple inductive limit algebra B of splitting tree algebras, homogeneous C*-algebras with base spaces wedge product of the Gong standard spaces and the two sphere S^2, such that A and B has the same K-theory, and the C*-algebra B satisfies the following.*

Let G_0 be a finite generated subgroup of $K_0(B)$ with decomposition $G_0 = G_{00} \oplus G_{01}$, where G_{00} is vanished by all states of $K_0(A)$. Suppose $\mathcal{P} \subset \underline{K}(B)$ is a finite subset which generates a subgroup G such that $G_0 \subset G \cap K_0(B)$.

Then for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset B$, any $1 > r > 0$, and any positive integer K, there is an $\mathcal{F} - \varepsilon$ multiplicative map $L : B \to B$ such that:

1. *$[L]|_{\mathcal{P}}$ is well defined.*
2. *$[L]$ induces the identity maps on the infinitesimal part of each of $G \cap K_0(B)$, $G \cap K_1(B)$, $G \cap K_0(B, \mathbb{Z}/k\mathbb{Z})$ and $G \cap K_1(B, \mathbb{Z}/k\mathbb{Z})$ (for k with $G \cap K_i(B, \mathbb{Z}/k\mathbb{Z}) \neq \{0\}$, $i = 0, 1$).*
3. *$\rho_B \circ [L](g) \leq r\rho_B(g)$ for all $g \in G \cap K_0(B)$.*
4. *For any positive element $g \in G_{01}$, we have $g - [L](g) = Kf$ for some $f \in K_0(B)^+$.*

Proof. Let us construct a C*-algebra B using the given building blocks such that the infinitesimal part of K_0-group only appears in finite stage building blocks.

Recall that ρ is the positive homomorphism from $\mathrm{K}_0(A)$ to $\mathrm{Aff}(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]_0)$. Denote by G the image $\rho(\mathrm{K}_0(A))$. Since a quotient group of the K_0-groups of a splitting tree algebra is still a K_0-groups of a splitting tree algebra, and the convex of states of G is canonically isomorphic to the convex of states of $\mathrm{K}_0(A)$, the ordered group G has the following property

For any finite subset $\mathcal{F} \subset G^+$, any $\varepsilon > 0$, there is a subgroup S of G which is isomorphic to the K_0-group of a splitting tree algebra such that for any $p \in \mathcal{F}$, we have a decomposition $p = p_1 + p_2$ such that $p_2 \in S$ and $r(p_1) < \varepsilon$ for any state r of G.

Since G is torsion free, by Corollary 3.4 of [6] or Theorem 8.2.1 of [23], we have an inductive limit decomposition $G = \varinjlim(G_i, \alpha_i)$ where G_i is a finite direct sum of K_0-group of a splitting tree algebra and $(\mathbb{Z}, \mathbb{Z}^+)$. By the proof of Corollary 3.4 of [6] or Theorem 8.2.1 of [23], we may assume that the G_i contains arbitrarily many copies of $(\mathbb{Z}, \mathbb{Z}^+)$ and the maps α_i restricted to $(\mathbb{Z}, \mathbb{Z}^+)$ has arbitrarily large multiplicity if i is sufficiently large.

Since $\ker \rho$ and $\mathrm{K}_1(A)$ are countable groups, we have that $\ker \rho = \varinjlim(H_i, \beta_i)$ and $\mathrm{K}_1(A) = \varinjlim(F_i, \iota_i)$ in the category of abelian groups where H_i and F_i are finite generated abelian groups. An argument same as that of Theorem 1.5 of [14] shows that the ordered group $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+)$ has a decomposition

$$(\mathrm{K}_0(A), \mathrm{K}_0(A)^+) = \varinjlim(G_i \oplus H_i, (G_i \oplus H_i)^+, \delta_n)$$

where the order on $G_i \oplus H_i$ is determined by the first coordinator, and the map δ_i has the form

$$\begin{pmatrix} \alpha_i & 0 \\ \gamma_i & \beta_i \end{pmatrix}.$$

Set $X_i = Y_1 \vee \cdots \vee Y_{n_i}$ where Y_i are S^2 or the Gong standard CW-complexes described in Section 2.3 such that $\mathrm{K}_0(\mathrm{C}(X_i)) = \mathbb{Z} \oplus H_i$ with the order determined by the first coordinator, and $\mathrm{K}_1(\mathrm{C}(X_i)) = F_i$.

Consider the ordered group G_i. Since it contains at least one copy of $(\mathbb{Z}, \mathbb{Z}^+)$, we can replace $(\mathbb{Z}, \mathbb{Z}^+)$ by the C*-algebra B_i and replace the remaining direct summand by the corresponding splitting tree algebras. Therefore, we have a C*-algebra $B_i = \mathrm{C}(X_i) \oplus S_i$ where S_i is a finite direct sum of splitting tree algebras such that $\mathrm{K}_0(B_i) = G_i$ and $\mathrm{K}_1(B_i) = F_i$. We may pass to matrix algebras of B_i such that $[1_{B_i}]_0 = u_i$ where $\delta_{i,\infty}(u_i) = [1_A]_0$. Thus, we may write $B_i = B_{i,1} \oplus B_{i,2}$ where $B_{i,1}$ is a matrix algebra over $\mathrm{C}(X_i)$ and $B_{i,2}$ is a finite direct sum of splitting tree algebras.

We show there is a *-homomorphism from B_i to B_{i+1} which lift the map δ_i. For the restrictions of the map δ_i to $B_{i,2}$ and $B_{i+1,2}$, these map can be lifted by Proposition 2.3. For the restrictions of δ_i to $B_{i,2}$ and $B_{i+1,1}$, since the K_0-group of $B_{i,2}$ does not contain infinitesimal elements, the map must factor through $\mathbb{Z}$, and therefore, by Proposition 2.3, it also can be lifted to a homomorphism. A same argument also apply to the restriction of δ_i to $B_{i,1}$ and $B_{i+1,2}$. The existence of a map lifting the restriction of δ_i to $B_{i,1}$ and $B_{i+1,1}$ follows an argument same as that of Theorem 1.5 of [14]. Denote by h_i the map from B_i to B_{i+1} which lifts δ_i.

By suitable choices of the point evaluation, the inductive limit B is simple. From the construction, one has

$$(\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B]) \cong (\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A])$$

and $\mathrm{K}_1(B) \cong \mathrm{K}_1(A)$, and moreover,

$$\ker \rho_{\mathrm{K}_0(A)} \cong \ker \rho_{\mathrm{K}_0(B)} = \varinjlim(\ker \rho_{\mathrm{K}_0(B_n)}, [h_n]_0).$$

Then, it is not difficult to see that the C*-algebra B satisfies the lemma (see the proof of Lemma 1.8 of [14]). □

Theorem 3.21. *Let A be a unital separable simple nuclear TAS-algebra which satisfies UCT. Suppose B is a simple inductive limit of splitting tree algebras and homogeneous algebras described in Lemma 3.20 with*

$$(K_0(A), K_0(A)^+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)^+, [1_B], K_1(B)).$$

Let $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B))^+$ implements this isomorphism between the K-invariants. Then for any finite subset $\mathcal{P} \subset P(A)$, there is a sequence of completely positive contractions $L_n : A \to B$ such that

1. *$\|L_n(ab) - L_n(a)L_n(b)\| \to 0$ for all $a, b \in A$ as $n \to \infty$, and*
2. *$[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for sufficiently large n.*

Proof. By Lemma 2.5, A is the closure of an increasing union of RFD C*-subalgebras $\{A_n\}$. We may assume $\mathcal{P} \subset \mathbf{P}(A_1)$. Let $\mathcal{P}_0 \subset \mathcal{P}$ such that $\mathcal{P}_0$ generate $G(\mathcal{P}) \cap \mathrm{K}_0(A)$ where $G(\mathcal{P})$ is the group generated by $\mathcal{P}$. Write $\mathcal{P}_0 = \{p_1, \cdots, p_l\}$ where p_i's are projections in a matrix algebra over A. Let $\mathcal{F}_1$ be a finite subset of A_1 and let $\delta_0 > 0$ be such that any $\mathcal{F}_1 - \delta_0$ multiplicative linear map L well-define $[L]|_{\mathcal{P}}$. Moreover, we requre $\mathcal{F}_1$ and δ_0 satisfy Lemma 3.17. Let k_0 be an integer such that $G(\mathcal{P}) \cap \mathrm{K}_i(A, \mathbb{Z}/k\mathbb{Z}) = \{0\}$ for any $k \geq k_0$, $i = 0, 1$.

By Theorem 3.19, there are two $\mathcal{F}_1 - \delta_0/2$ multiplicative completely positive linear maps Φ_0, Φ_1 from A to $B \otimes \mathcal{K}$ such that

$$[\Phi_0]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} + [\Phi_1]|_{\mathcal{P}}.$$

Φ_1 is a homomorphism when it is restricted on A_1, and the image is a finite dimensional C*-algebra. With Φ_1 in the role of h_0, we can proceed with the construction as described at the beginning of this section. We will keep the same notation.

Consider the map $\tilde{\rho} : G(\mathcal{P}) \cap \mathrm{K}_0(A) \to l^\infty(\mathbb{Q})$ defined in Remark 3.16. The linear span of $\{\tilde{\rho}(p_1), \cdots, \tilde{\rho}(p_l)\}$ over $\mathbb{Q}$ will have finite rank, say r. So, we may assume that $\{\tilde{\rho}(p_1), \cdots, \tilde{\rho}(p_r)\}$ are independent and the $\mathbb{Q}$-linear span of them give us the whole subspace. Therefore, there is an integer M such that for any $g \in \tilde{\rho}(G_0)$, Mg will be inside the subgroup generated by $\{\tilde{\rho}(p_1), \cdots, \tilde{\rho}(p_r)\}$. Let $x_{ij} = (\tilde{\rho}(p_i))_j$, and $z_i = \rho_A([p_i]) \in \mathbb{D}$, where $\mathbb{D} = \rho_A(\mathrm{K}_0(A))$ in $\mathrm{Aff}(S(K_0(A), 1))$. By Lemma 2.7, $\mathbb{D}$ is a dense subgroup of $\mathrm{Aff}(S(\mathrm{K}_0(A), 1))$. Note that $a_j^{(n)} \in \mathbb{D}^+ \backslash \{0\}$, $\lim_{n\to\infty} a_j^{(n)} = a_j > 0$ uniformly, and $\sum_{j=1}^n x_{ij} a_j^{(n)} \to z_i$ uniformly. So, Lemma 3.18 applies. Fix K and δ obtained from Lemma 3.18. Also note that if $g \in \ker \tilde{\rho}$, then $\tau(g) = 0$ for all traces τ, $g \in \ker[H]$ and $g \in \ker[\Phi_1]$. Hence g will be in the kernel of any direct sum of $[\Phi_1]$.

Let $\Psi := \Phi_0 \oplus (\underbrace{\Phi_1 \oplus \cdots \oplus \Phi_1}_{MK^3(k_0+1)!-1 \text{ copies}})$. Since Φ_1 factors through a finite dimensional C*-algebra, it will be zero when restricted to $\mathrm{K}_1(A) \cap G$, $\mathrm{K}_1(A_1, \mathbb{Z}/k\mathbb{Z}) \cap G$. Moreover, $(\underbrace{\Phi_1 \oplus \cdots \oplus \Phi_1}_{MK^3(k_0+1)! \text{ copies}})$ induces the zero map on $\mathrm{K}_0(A, \mathbb{Z}/k\mathbb{Z})$. Therefore we have

$$[\Psi]|_{K_1(A)\cap G} = \alpha|_{K_1(A)\cap G}, \quad [\Psi]|_{K_1(A,\mathbb{Z}/k\mathbb{Z})\cap G} = \alpha|_{K_1(A,\mathbb{Z}/k\mathbb{Z})\cap G}$$

and $[\Psi]|_{K_0(A,\mathbb{Z}/k\mathbb{Z})\cap G} = \alpha|_{K_0(A,\mathbb{Z}/k\mathbb{Z})\cap G}$. We may assume $\Psi(1_A)$ is a projection in $\mathrm{M}_r(B)$ for some r. As we can see, the map Ψ induces the desired maps on all the invariants except the K_0 group. In the following part of the proof, we are going to compress the map Ψ to a small corner of $B \otimes \mathcal{K}$, and then find a suitable map h to fix Ψ on the K_0 part.

We may assume there exist projections $\{p_1', \cdots, p_l'\}$ inside $B \otimes \mathcal{K}$, which are sufficiently closed to $\{\Psi(p_1), \cdots, \Psi(p_l)\}$ respectively. So, we get $[p_i'] = [\Psi(p_i)]$. Note that B itself is a simple TAS-algebra, and therefore the strict order on the projections of B is determined by traces. Thus we can find a subprojection q_i' inside each p_i', such that $[q_i'] = MK^3(k_0+1)![\Phi_0(p_i)]$. Set $e_i' = p_i' - q_i'$, and let $\mathcal{P}_1 = \Psi(\mathcal{P}) \cup \Phi_1(\mathcal{P}) \cup \{p_i', q_i', e_i'; i = 1, \cdots, l\}$. Set G_1 to be the group generated by $\mathcal{P}_1$. Remember that $G_0 = G(\mathcal{P}) \cap \mathrm{K}_0(A)$. It can be written as $G_{00} \oplus G_{01}$, where G_{00} is the infinitesimal part of G_0. Let $\{d_1, \cdots, d_t\}$ be positive elements which generate G_{01}.

Now applying Theorem 3.20 to $\mathrm{M}_r(B)$ with any finite subset $\mathcal{G}$, any $\varepsilon > 0$ and any $0 < r_0 < \delta < 1$, we get a $\mathcal{G} - \varepsilon$-multiplicative map $L : \mathrm{M}_r(B) \to \mathrm{M}_r(B)$ with the following properties:

(1) $[L]|_{\mathcal{P}_1}$ and $[L]|_{G_1}$ are well defined;
(2) $[L]$ induces the identity maps on the infinitesimal part of $G_1 \cap \mathrm{K}_0(B)$, $G_1 \cap \mathrm{K}_1(B)$, $G_1 \cap \mathrm{K}_0(B, \mathbb{Z}/k\mathbb{Z})$ and $G_1 \cap \mathrm{K}_1(B, \mathbb{Z}/k\mathbb{Z})$ for the k with $G_1 \cap \mathrm{K}_i(B, \mathbb{Z}/k\mathbb{Z}) \neq \{0\}$, $i = 0, 1$;
(3) $\tau \circ [L](g) \le r_0 \tau(g)$ for all $g \in G_1 \cap \mathrm{K}_0(B)$ and $\tau \in T(B)$;
(4) There exist positive elements $\{f_i\} \subset \mathrm{K}_0(B)^+$ such that for $i = 1, \cdots, t$,

$$\alpha(d_i) - [L](\alpha(d_i)) = MK^3(k_0+1)! f_i.$$

We can choose r_0 is sufficient small such that $\tau \circ [L] \circ [\Psi]([p_i]) < \delta/2$ for all $\tau \in T(B)$, and $\alpha([p_i]) - [L \circ \Psi]([p_i]) > 0$. This can be done since $T(B)$ is compact and B has the fundamental comparison property of Blackadar.

Let $[p_i] = \sum m_j^{(i)} d_j + s_i$ where $m_j^{(i)} \in \mathbb{Z}$ and $s \in G_{00}$. Then we have

$$\begin{aligned} & \alpha([p_i]) - [L \circ \Psi]([p_i]) \\ = \; & \alpha(\sum m_j^{(i)} d_j) - ([L \circ \alpha](\sum m_j^{(i)} d_j) + MK^3(k_0+1)![L \circ \Phi_1]([p_i])) \\ = \; & (\alpha(\sum m_j^{(i)} d_j) - [L \circ \alpha](\sum m_j^{(i)} d_j)) - MK^3(k_0+1)![L \circ \Phi_1]([p_i]) \\ = \; & MK^3(k_0+1)!(\sum m_j^{(i)} f_j - [L] \circ [\Phi_1]([p_i])) \\ = \; & MK^3(k_0+1)! f'_j, \quad \text{if we write} \quad f'_j = \sum m_j^{(i)} f_j - [L] \circ [\Phi_1]([p_i]). \end{aligned}$$

Since $\alpha([p_i]) - [L \circ \Psi])([p_i]) > 0$ and $\mathrm{K}_0(B)$ is weakly unperforated, $f'_j > 0$. Let us set $\beta : G(\mathcal{P}) \cap \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ by $\beta([p_i]) = K^3(k_0+1)! f'_i$.

Now we are ready to find a map h' from A to B which will carry the map β. It will be constructed by factoring through some splitting interval algebras in the construction given at the beginning of this section. Let $\tilde{z'_i} = \beta([p_i])$, and $z'_i = \rho_B(\tilde{z'_i}) \in \mathrm{Aff}(\mathrm{K}_0(B), 1)$. Recall that we identify the $\mathrm{K}_0(A)$ and $\mathrm{K}_0(B)$ and α is the identity. Then we have:

$$\begin{aligned} \|Mz' - z\|_\infty &= \max_i\{\|\rho(\alpha([p_i]) - [L \circ \Psi]([p_i])) - \rho([p_i])\|\} \\ &= \max_i\{\sup_{\tau \in T(B)}\{\tau \circ [L] \circ [\Psi]([p_i])\} \\ &\leq \delta/2. \end{aligned}$$

By Lemma 3.18, we get $\tilde{u} = \{\{u_1^{(1)}, \cdots, u_{r_1}^{(1)}\}, \cdots, \{\{u_1^{(s(n))}, \cdots, u_{r_{s(n)}}^{(s(n))}\}\}$ where $u_i^{(j)}$'s are positive elements of $\mathrm{K}_0(B)$ such that $\sum x_{ij} u_j = \tilde{z}_i$. Moreover, $\{u_k^{(k)}, \cdots, u_{r_k}^{(k)}\}$ will satisfy the same relations as the positive generators of $\mathrm{K}_0(S_k)$. Therefore, if we let $D = S_1 \oplus \cdots \oplus S_{s(n)}$, there will be an ordered homomorphism from $\mathrm{K}_0(D)$ to $\mathrm{K}_0(B)$ such that send the positive generator of $\mathrm{K}_0(S_k)$ to the correspondence $u_j^{(k)}$. Since B is a inductive limit of splitting tree algebras together with homogeneous C*-algebras, by the proposition 2.3, this K-theory map can be lifted to a $*$-homomorphism $h' : D \to \mathrm{M}_k(B)$. So, we have

$$[h'](\tilde{\rho}([p_i])) = \beta([p_i]), \quad i = 1, \cdots, r,$$

if we look at $\tilde{\rho}([p_i])$ being truncated into D. Now, we can consider h' as a map from A to $\mathrm{M}_k(B)$ by composing it with the map $\psi_1 \oplus \cdots \oplus \psi_{s(n)} : A \to D$. It is $\mathcal{F} - \delta$ multiplicative.

For any $x \in \ker \tilde{\rho}$, by Lemma 3.17, $x \in \ker \tau \circ \alpha \cap \ker[H]$ and $x \in \ker[h_0] = \ker[\Phi_1]$. Therefore, we have $[\Phi_1](x) = 0$ and $[\Psi](x) = \alpha(x)$. Note that $\alpha(x)$ also vanishes under any state of $(\mathrm{K}_0(B), \mathrm{K}_0^+(B))$, we have $[L] \circ \alpha(x) = \alpha(x)$. So, we get

$$\alpha(x) - [L \circ \Psi](x) = 0.$$

Therefore $\alpha - [L \circ \Psi]$ gives us a homomorphism on $\tilde{\rho}(G_0)$.

Set h to be M copies of h'. h will also be $\mathcal{F} - \delta$ multiplicative, and

$$[h]([p_i]) = \alpha([p_i]) - [L] \circ [\Psi]([p_i]) \quad i = 1, \cdots, l.$$

Note that $[h]$ also has the multiplicity $MK^3(k_0+1)!$, and D is a splitting interval algebra (which has trivial K_1 groups). One can conclude that h induces zero map on $G \cap \mathrm{K}_1(A)$, $G \cap \mathrm{K}_1(A, \mathbb{Z}/k\mathbb{Z})$

and $G \cap \mathrm{K}_0(A, \mathbb{Z}/k\mathbb{Z})$ for $k \leq k_0$. Therefore, we have

$$[h]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} - [L] \circ [\Psi]|_{\mathcal{P}}.$$

Set $L_1 = L \circ \Psi \oplus h$. It is $\mathcal{F} - \delta$ multiplicative and

$$[L_1]|_{\mathcal{P}} = [h]|_{\mathcal{P}} + [L] \circ [\Psi]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

We may assume $L_1(1_A) = 1_B$ by taking a conjugation with a partial isometry. Then L_1 is a map from A to B, and gives us the desired K-theory map. □

3.3. **Local existence theorem for maps between two TASI-algebras.** In this subsection, we are going to study the C*-algebras which can be tracially approximated by splitting interval algebras (TASI-algebras for short). We are going to lift a given invariant map approximately, not only for the given positive KL-element, but also for the map on the simplex of traces. First, we show that for any KK-attainable TAS-algebra, one can always modify the maps to give us the compatible trace map simultaneously. More precisely, we have:

Proposition 3.22. *Let A be a KK-attainable TAS-algebra, and let B be a simple TASI-algebra. Then for any $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B))^+$, and any $\Gamma : T(B) \to T(A)$ which is compatible with α, there exist a sequence of completely positive linear contractions $\{L_n\}$ which satisfy the conditions in Definition 3.4, and they also almost induce the trace map Γ, i.e.,*

$$\sup_{\tau \in T(B)} \{|\Gamma(\tau)(a) - \tau \circ L_n(a)|\} \to 0 \quad \forall a \in A.$$

Proof. Let $\mathcal{F}$ be any finite subset of A, $\mathcal{P}$ be any finite subset of $P(A)$ contains $[1_A]$ and any $\varepsilon > 0$. We may assume $\alpha([1_A]) = [1_A]$ and $\mathcal{F}$ is large enough so that contains all the entries of the projections in $\mathcal{P}$. Since A is a TAS-algebra, for any finite subset $\mathcal{F}' \subset A$ and $\delta > 0$, we can find a subalgebra S which is a splitting interval algebra with $p = 1_S$ such that $\phi : a \mapsto pap$ and $\psi : a \mapsto (1-p)a(1-p)$ are $\mathcal{F}' - \delta$ multiplicative and $\tau(1_A - p) \leq \delta$. Therefore, with sufficiently large $\mathcal{F}'$ and sufficiently small δ, we may assume that $[\phi]|_{\mathcal{P}} + [\psi]_{\mathcal{P}} = \mathrm{id}|_{\mathcal{P}}$ and $\phi(a) \in S$ for any $a \in \mathcal{F}$.

Let $\mathcal{S}$ be a set of the generators of S and set $\mathcal{G} = \mathcal{F} \cup \mathcal{S}$. Since A is KK-attainable, there is a sequence of completely positive linear contractions $L_n : A \to B$ such that for any $a, b \in A$,

$$||L_n(ab) - L_n(a)L_n(b)|| \to 0,$$

and for sufficiently large n,

$$[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

Therefore, for any $\varepsilon_0 > 0$, there is a $\mathcal{G} - \varepsilon_0$ multiplicative linear contraction L_n such that

$$[L_n \circ \phi]|_{\mathcal{P}} + [L_n \circ \psi]_{\mathcal{P}} = [L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

Denote by L'_n the restriction of $L_n \circ \phi$ to S. Since the generators of S satisfy stable relations, we may assume L' is a *-homomorphism. Set $\mathcal{G}'$ to be $L_n(\mathcal{G})$. Since B is also a TAS-algebra, by the same argument as for A, we can find a C*-subalgebra S' of B which is a splitting interval algebra with identity q, such that $\tau(1-q) \leq \varepsilon$ for any trace $\tau \in B$ and

$$[qL'q]|_{\mathcal{P}} + [(1-q)L'(1-q)]|_{\mathcal{P}} = [L']|_{\mathcal{P}}.$$

Moreover, we may assume $qL'q$ is a $*$-homomorphism from S to S'. Set κ to be the K_0 map induced by qL'_nq. Furthermore, there also exist a trace map θ' from $T(S')$ to $T(B)$ such that $|\theta'(\tau)(a)-\tau(a)|\leq$ for any $a\in L'_n(\mathcal{F})$. Then $\theta=\frac{1}{1-\varepsilon}\Gamma\circ\theta'$ will give us an affine map from $T(S')$ to $T(S)$. So, we have

$$\theta'(\tau)(\alpha(p))=\theta'(\tau)([L'_n](p)+[(1-q)L_n(p)(1-q)]),$$

for any $\tau\in T(S')$ and any projection $p\in S$. On the other hand, since Γ and α are compatible, we have

$$\begin{aligned}\Gamma\circ\theta'(\tau)(p) &= \theta'(\tau)(\alpha(p))\\ &= \theta'(\tau)([L'_n](p)+[(1-q)L_n(p)(1-q)]),\end{aligned}$$

for any $\tau\in T(S')$ and any projection $p\in S$. Hence we have

$$\begin{aligned}|\tau(\kappa(p))-\theta(\tau)(p)| &\leq |\theta'(\tau)(L'_n(p))-\tfrac{1}{1-\varepsilon}\Gamma\circ\theta'(\tau(p))|+\varepsilon\\ &\leq |\Gamma\circ\theta'(\tau)(p)-\tfrac{1}{1-\varepsilon}\Gamma\circ\theta'(\tau(p))|+2\varepsilon\\ &\leq 3\varepsilon.\end{aligned}$$

for any K_0 class of the projections in S and any trace τ over S'. Then by the existence theorem in [9], there is a $*$-homomorphism $h: S\to S'$ such that $h_*=\kappa$ and $|\tau\circ h(a)-\theta(\tau)(a)|\leq\varepsilon$ for any $a\in p\mathcal{F}p$ and $\tau\in T(A)$. For the convenience, we still denote the tolerant to be ε. Now, let $L=L_n\circ\psi+(1-q)L'(1-q)+h\circ\phi$. It satisfies

$$[L]|_{\mathcal{P}}=\alpha|_{\mathcal{P}},$$

and

$$|\tau\circ L(a)-\Gamma(\tau)(a)|<6\varepsilon,$$

for any $a\in\mathcal{F}$ and any trace τ on B. Thus, the proposition is proved. ☐

The following proposition gives us a class of KK-attainable TAS-algebras.

Proposition 3.23. *Any simple separable nuclear TAS-algebra satisfying the UCT is KK-attainable.*

Proof. Let A be a simple nuclear TAS-algebra satisfying UCT. By Lemma 3.20, there is a simple inductive limit of splitting tree algebra, together with the Gong's standard homogeneous C*-algebras and homogeneous C*-algebras with base space S^2 such that

$$((\mathrm{K}_0(A),\mathrm{K}_0(A)^+,[1_A]),\mathrm{K}_1(A))\cong((\mathrm{K}_0(C),\mathrm{K}_0(C)^+,[1_C]),\mathrm{K}_1(C)).$$

Since A satisfies the UCT, there is $\beta\in\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A),\underline{\mathrm{K}}(C))^+$ which induces the isomorphism between the invariants of A and C. Moreover, we can choose β to be invertible. Note that the C*-algebra C is KK-attainable by Remark 3.12.

For any simple TAS-algebra B, any $\alpha\in\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A),\underline{\mathrm{K}}(B))^+$, and any finite subset $\mathcal{P}\subset P(A)$, we consider the KL-element $\alpha\beta^{-1}\in\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(C),\underline{\mathrm{K}}(B))^+$, and the subset $\mathcal{P}'\subset P(C)$ which presents $\alpha(\mathcal{P})$. Since C is KK-attainable, there exist a sequence of completely positive linear contractions (L_n) from C to B such that (L_n) are approximately multiplicative, and $[L_n]|_{\mathcal{P}'}=\alpha\beta^{-1}|_{\mathcal{P}'}$ for sufficiently large n.

By Theorem 3.21, there is a sequence of approximately multiplicative completely positive linear contractions (L'_n) from A to C such that L'_n induces β on $\mathcal{P}$ for sufficiently large n.

Then the compositions $L_n \circ L'_m : A \to B$ with suitable choice of m, n will give us a sequence of completely positive linear contractions which are approximately multiplicative, and $[L_n \circ L'_m]|_{\mathcal{P}} = (\alpha \circ \beta) \circ \beta^{-1}|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for sufficiently large m, n. Therefore, A is KK-attainable. □

Corollary 3.24. *Let A and B be two simple TASI-algebras satisfying UCT. Then for any $\alpha \in Hom_{\Lambda}(\underline{K}(A), \underline{K}(B))^+$ with $\alpha([1_A]) = [1_B]$, any finite subset $\mathcal{P} \in P(A)$, and any trace map $\theta : T(B) \to T(A)$ which is compatible with α, there is a sequence of completely positive linear contractions $L_n : A \to B$ such that*

(1) $\|L_n(ab) - L_n(a)L_n(b)\| \to 0$ *for any $a, b \in A$;*
(2) $[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ *for sufficient large n;*
(3) $|\theta(\tau)(a) - \tau(L_n(a))| \to 0$ *for any $a \in A$ and $\tau \in T(B)$.*

Proof. This corollary follows directly from the proposition above and Proposition 3.22. □

4. Uniqueness Theorem

In this chapter, we are going to establish the uniqueness theorem for TASI-algebras. The strategy is to get a stable uniqueness theorem for TASI-algebras first. Then, using an approximately divisibility property of TASI-algebras, one can decompose a map between two TASI-algebras into a map with large multiplicity. Thus one can use the stable uniqueness to show the uniqueness of the original map.

First, we show that any element in a TASI-algebra can be approximated by the sum of an element with prescribed multiplicity and an element with small trace. More precisely, we show the following theorem.

Theorem 4.1. *Let A be a TASI-algebra, and let $\mathcal{F}$ be a finite subset of A. Then for any natural number n, any $\varepsilon > 0$, there exist mutually orthogonal projections $q, p_1, \cdots, p_n$ with $q + p_1 + \cdots + p_n = 1$, $q \preceq p_1$ and $p_i \sim p_1$, a splitting interval sub-C*-algebra S with $1_S = p_1$ and two $\mathcal{F} - \varepsilon$ multiplicative linear unital maps $L_0 : A \to qAq$, $L_1 : A \to S$, such that*

$$||x - L_0(x) \oplus (\underbrace{L_1(x) \oplus \cdots \oplus L_1(x)}_{n \quad copies})|| \le \varepsilon,$$

and $\tau(q) \le \varepsilon$ for all trace of A.

Proof. Since A is a TASI-algebra, for any $\varepsilon_0 > 0$, we can find a splitting interval algebra S inside A with $p' = 1_A$ such that

(1) $\|[x, p']\| \le \varepsilon_0$ for any $x \in \mathcal{F}$,
(2) $p'xp' \in_{\varepsilon_0} S$ for any $x \in \mathcal{F}$,
(3) $\tau(1 - p') < \varepsilon_0$ for any trace τ of A.

Set $\mathcal{F}'$ to be the union of $\mathcal{F}$ and a generating set for S. By applying the tracial approximation, we get another splitting interval algebra S' inside A such that $\tau(1 - 1_{S'}) \le \varepsilon_0$ for all traces of A, and a homomorphism ϕ from S to S' which is approximately the cut down by $1_{S'}$ on $\mathcal{F}'$. Set $p_0 = \phi(p')$. One can see that $\tau(1 - p_0) < 2\varepsilon_0$ and $S_0 = p_0S'p_0$ is a splitting interval algebra. What is more, since A is simple, by an asymptotic argument, we may assume that the size of each canonical quotien of S_0 is large enough and the eigenvalue maps of ϕ are relatively dense, i.e., for any point x_0 in the spectrum of S_0, the evaluation points of $\phi \circ e_{x_0}$ are δ-dense in the spectrum of S. Let M be the size of the matrix at the regular points of S, and k be the number of the splitting points of S.

Denote the eigenvalue maps by $\{\lambda_1, \cdots, \lambda_m\}$. Each of these is a continuous map from the solid interval (the canonical quotient of the spectrum of S_0) to the spectrum of S. If some λ_i maps into a single splitting point of S, it must be a constant map (the point evaluation of that splitting point). For all the other eigenvalue maps, we view them as maps to the solid interval. Then as in interval algebras case, these eigenvalue maps can be perturbed and rearranged so that they do not cross with each other. Since $\mathcal{F}$ is a finite subset, we may assume the eigenvalue maps of ϕ have a such form.

For the regular eigenvalue maps, we will perturb them more if they are around the singular point of S: if it is close to a singular point up to δ_0, we will replace them by the point evaluation on that singular point (we are allowed to do this because the only singular points of S are 0 and 1, and the eigenvalue maps are almost constant around this two points). We assume that δ_0 is small enough. By applying the asymptotic argument on $e_x \circ \phi$ for each splitting point of S_0, we see that the multiplicity of each singular point of S in x is sufficiently large (since the number of the eigenvalues in any intervals with length δ_0 is sufficiently large), say bigger than M_0. For the rest of m' regular eigenvalue maps, we will group each consecutive $[m'\delta_0]$ eigenvalue maps together, and replace all of them by one eigenvalue map in that group. Since m' could be sufficiently large and they are relatively dense, this procedure does not change the eigenvalue maps too much. And by the same argument as above, we also assume the multiplicity of each group on any singular points of S_0 is larger than M_0. Therefore, each eigenvalue map has at least multiplicity M_0, and the map ϕ can be written as

$$\phi : f \mapsto \sum_{i=1}^{k+[1/\delta_0]} (f \circ \lambda_i)p'_i \quad \forall f \in \mathcal{F},$$

where $p'_i = p''_{i,1} + \cdots + p''_{i,m_i}$ for a collection of mutually orthogonal projections $\{p''_{i,j}\}$ with $p''_{i,j} \sim p''_{i,1}$ for each j and $m_i \geq M_0$.

Set $N = k + 1/\delta_0$ to be number of the distinguished eigenvalue maps, and take M_0 to be larger than $(n \cdot N)/\varepsilon_0$. Then $m_i = ns_i + r_i$ for some $0 \leq r_i < n$. We can divide p'_i into $n+1$ groups with each of the first n groups consists s_i small projections, i.e., $p_j^{(i)} = p''_{i,js_i+1} + \cdots + p''_{i,js_i+s_i}$ for $j = 1, \cdots, n$ and $q^{(i)} = p''_{i,ns_i+1} + \cdots + p''_{i,m_i}$. Notice that $\tau(q^{(i)}) \leq n\tau(p''_{i,1}) \leq \varepsilon_0/N$. Moreover, we set

$$p_j = p_j^{(1)} + \cdots + p_j^{(k)}.$$

One can see that p_j's are mutually orthogonal, and Murray-von Neumann equivalent to each other. Let $q = 1 - \sum p_j$. Then for any trace τ of A, we have

$$\begin{array}{rcl}\tau(q) & = & \tau(1 - p_0 + \sum_1^N q^{(i)}) \\ & \leq & 2\varepsilon_0 + \varepsilon_0.\end{array}$$

So, we can set

$$L_0 : x \mapsto qxq$$

and

$$L_1 : x \to \phi'(q'xq')$$

where $\phi' : f \mapsto \sum_{i=1}^{k+[1/\delta_0]}(f \circ \lambda_i)p_1^{(i)}$ is a fraction of ϕ. These two maps are $\mathcal{F} - \varepsilon$ multiplicative, and

$$||x - L_0(x) \oplus (\underbrace{L_1(x) \oplus \cdots \oplus L_1(x)}_{n \text{ copies}})|| \leq \varepsilon.$$

Moreover, we can let ε_0 to be sufficient small such that $\tau(q) < 1/(2n) < \tau(p_1)$ for all trace τ. Then $q \preceq p_1$ as desired. □

As a corollary, we can show the for any finite subset of a TAS-algebra, we can find a finite dimensional C*-subalgebra with arbitrarily large size simple summand, and it almost commutes the large cut-down of the given finite subset.

Corollary 4.2. *Let A be a TAS-algebra, and $\mathcal{F} \subset A$ be a finite subset. Then for any $\varepsilon > 0$, any natural number N, there exists a finite dimensional sub-C*-algebra B witth $p = 1_B$ such that*

1. *the size of each simple component of B is bigger than N;*
2. *$\|[p, x]\| \leq \varepsilon$ for any $x \in \mathcal{F}$;*
3. *$\|[pxp, b]\| \leq \varepsilon$ for any $x \in \mathcal{F}$ and $b \in B$ with $\|b\| \leq 1$;*
4. *$\tau(1 - p) \leq \varepsilon$ for any trace τ on A.*

Proof. By the theorem above, there exist projections $q, p_i, i = 1, \cdots, N$, and $\mathcal{F} - \delta$ multiplicative maps L_0, L_1 such that:

$$||x - L_0(x) \oplus (\underbrace{L_1(x) \oplus \cdots \oplus L_1(x)}_{N \text{ copies}})|| \leq \varepsilon.$$

and $\tau(q) \leq \varepsilon$ for all $\tau \in T(A)$.

Then the projections p_i's and the partial isometries between them will generate an $N \times N$ matrix sub-algebra of A. Denote it by B. It commutes with $L_1(x) \oplus \cdots \oplus L_1(x)$. Let p to be its identity $p = p_1 + \cdots + p_N$, we see that $\tau(1 - p) = \tau(q) \leq \varepsilon$. Thus B is the desired finite dimensional C*-subalgebra in the corollary. □

Proposition 4.3. *Let S be a splitting tree algebra, A be a simple TAS-algebra. Then, for any finite subset $\mathcal{F} \subset S$, any $\varepsilon > 0$, there is a finite subset $\mathcal{G} \subset S$ and $\delta > 0$ such that for any two homomorphisms ϕ and ψ from S to A, if*

1. *$\phi_* = \psi_*$ on $K_0(S)$ and*
2. *$\|\tau \circ \phi(g) - \tau \circ \psi(g)\| < \delta$ for any $g \in \mathcal{G}$ and $\tau \in T(B)$,*

then there exists a unitary $u \in A$ such that

$$\|\phi(f) - u\psi(f)u^*\| < \varepsilon \text{ for all } f \in \mathcal{F}.$$

Proof. Assume S is the full $k \times k$ matrix at the regular points, and $\mathcal{F}$ is a generating set of S which contains the central elements $\{t(1-t)e_{ij}\}$.

First, we show that ϕ and ψ can be decomposed approximately (on $\mathcal{F}$) as a direct sum of a homomorphism together with many point evaluation maps. To find one point evaluation map of S, it is enough to find a system of $k \times k$ matrix units in A on which $\phi(S)$ (or $\psi(S)$) acts as a point evaluation. For any $n \in \mathbb{Z}^+$, let $\xi_i = \frac{i}{n}$, and take $\{s_{i,j}\}_{i=0}^n$ to be the elements of S which are 0 outside $(\frac{2i-1}{2n}, \frac{2i+1}{2n})$, e_{jj} on $[\frac{4i-1}{4n}, \frac{4i+1}{4n}]$, and linear between. Set $s_i = \sum_j s_{i,j}$. Then $\{\phi(s_{i,j})\}$ are mutually orthogonal since $\{s_{i,j}\}$ are mutually orthogonal. Take $p_{i,j}$ to be a non-zero projection inside the hereditary C*-subalgebra of A which is generated by $\phi(s_{i,j})$. This can be done since A has the (SP) property. Moreover, since A is simple, we can assume $p_{i,j}$'s are equivalent to each others. Set $p_i = \sum_j p_{i,j}$. Note that $fs_i - f(\xi_i)s_i < (1/n)s_i$ for any $f \in \mathcal{F}$; therefore, we have $\|\phi(f)p_i - f(\xi_i)p_i\| < 1/n$. Set $e = \sum p_i$, we also have $\|\phi(f)e - f(\xi_i)e\| < 1/n$. Hence we may write ϕ (on $\mathcal{F}$) as

$$\phi(f) = (1-e)\phi(f)(1-e) + \sum f(\xi_i)p_i.$$

By the same argument, we also get

$$\psi(f) = (1-e')\psi(f)(1-e') + \sum f(\xi_i)q_i,$$

where $e' = \sum q_i$. Since A is simple and has the (SP) property, we may assume that $p_i = q_i$ and the p_i's are Murray-von Neumann equivalent to each other (by passing to the subprojections and composing inner automorphisms). Moreover, since the positive cone of $\mathrm{K}_0(S)$ is finite generated, we may assume $(1-e)\phi(1-e)$ and $(1-e)\psi(1-e)$ induce same map on $\mathrm{K}_0(S)$.

For any two projections $p, q \in S$ which are not Murray-von Neumann equivalent to each other, if $[(1-e)\phi(p)(1-e)] = [(1-e)\psi(q)(1-e)]$, then there exists a partial isometry $v \in (1-e)A(1-e)$ which induces the equavilence relation. Pick one such partial isometry v for each pair of minimal projections which have equivalent images. Denote all such partial isometries v by V. Since there are only finite minimal projections in S, we get that V is a finite subset of A. Let $\mathcal{F}' = \phi(\mathcal{F}) \cup \psi(\mathcal{F}) \cup V$. Since $(1-e)A(1-e)$ is a TAS-algebra, there is a projection p and a splitting tree algebra $S' \subset (1-e)A(1-e)$ with $1_{S'} = p$ such that

(1) $\|px - xp\| < \varepsilon_0$,
(2) $pxp \in_{\varepsilon_0} S'$ for all $x \in \mathcal{F}'$, and
(3) $1 - p \preceq p_{11}$.

Since ε_0 can be arbitrarily small, we may assume ϕ and ψ are the direct sum of two homomorphisms:

$$\phi(f) = \phi'(f) + (\phi''(f) + \sum f(\xi_i)p_i)$$

$$\psi(f) = \psi'(f) + (\psi''(f) + \sum f(\xi_i)p_i)$$

for any $f \in \mathcal{F}$, where $\phi', \phi'', \psi', \psi''$ are the cut-down of ϕ and ψ by p and $1-p$ respectively.

The maps ϕ' and ψ' are homomorphisms from S to S'. Since $\mathrm{K}_0(S)$ is finite generated by the minimal projections, we conclude that $\phi'_* = \psi'_*$ on $\mathrm{K}_0(S)$. What is more, we may assume that $\|\tau \circ \phi'(g) - \tau \circ \psi'(g)\| < \delta$ for all $g \in \mathcal{G}$, and S' has large multiplicity by the simplicity of A. By [9], there is an unitary $u_1 \in S' \subset pAp$ such that $\|\phi'(f) - u_1\phi'(f)u_1^*\| < \varepsilon$ for any $f \in \mathcal{F}$.

Now we consider the map $f \mapsto \phi''(f) + \sum_{i=1}^{n} f(\xi_i)p_i$, and show that it can be approximated by point evaluation maps. Let S'' be the image of S under ϕ''. Since splitting interval algebras have stable generators and relations, we may assume S'' is a splitting interval algebra. Since $\phi''(1_S) \preceq p_{11}$, we have that S'' is isomorphic to a C*-subalgebra of $p_{11}Ap_{11}$ via a partial isometry of A. But all p_{ij}'s are Murray-von Neumann equivalent, so by passing to the subprojections of p_i's, we may assume that the map $f \mapsto \phi''(f) + \sum_{i=1}^{n} f(\xi_i)p_i$ is a map from S to the $(nk+1) \times (nk+1)$ matrix algebra of S'', where k is the generic dimension of S''. Thus we can consider the corresponding eigenvalue maps between S and its image. By the constructions, almost all the eigenvalue maps of $f \mapsto \phi''(f) + \sum_{i=1}^{n} f(\xi_i)p_i$ are constant maps (with sufficiently large n). Hence by [26], we can perturb all the eigenvalue maps to constant maps. Therefore, there is a homomorphism $\tilde{\phi}'' : S \to \mathrm{M}_{nk+1}(S'')$ with constant eigenvalue maps (in other words, it comes from point evaluations) such that

$$\left\| \tilde{\phi}''(f) - u'(\phi''(f) + \sum_{i=1}^{n} f(\xi_i)p_i)u'^* \right\| < \frac{1}{n} \quad \forall f \in \mathcal{F}$$

with some unitary $u' \in (1-p)A(1-p)$. So, we may assume $f \mapsto \phi''(f) + \sum_{i=1}^{n} f(\xi_i)p_i$ is a points evaluation map. Denote it by $\tilde{\phi}''$. Note that the evaluated points are relative dense up to $1/n$ in $(0, 1)$.

By the same argument, we also may assume that $f \mapsto \psi''(f) + \sum_{i=1}^{n} f(\xi_i)p_i$ is a sum of points evaluation maps, denote it by $\tilde{\psi}''$. Now, we are going to pair their evaluated points (constant eigenvalue maps) by the K_0 maps they induced.

By adjoining the unitary, we may assume that the images of $\tilde{\phi}''$ and $\tilde{\psi}''$ share same finite dimensional C*-subalgebra of A. Furthermore, we also can assume that the point evaluations on the singular point is not full by moving the full points evaluation maps on the singular points to the regular point nearby slightly. First we conclude that $\tilde{\phi}''$ and $\tilde{\psi}''$ has the same multiplicity on the regular points. To see this, set p to be a minimal projection of S such that p is 0 on the spitting points which $\tilde{\phi}''$ take point evaluation. Then we have $[\tilde{\phi}''(p)] = m[d]$ and $[\tilde{\psi}''(p)] = (n+k)[d]$ for some $k \geq 0$, where d is a minimal projection of the matrix algebra, m and n are the multiplicities of $\tilde{\phi}''$ and $\tilde{\psi}''$ respectively. But since $\tilde{\phi}''_* = \tilde{\psi}''_*$, we get $m \geq n$ after applying a trace on both sides (A is simple, so $\tau(d) \neq 0$). The same argument also shows that $m \leq n$. This gives us the desired conclusion. Since we can choose n is large enough such that the evaluated points has the "δ-density" as in interval algebras case [10], we can find a pairing between the regular evaluation points of $\tilde{\phi}''$ and $\tilde{\psi}''$.

Now we consider the splitting points. Let us denote the splitting points of S to be $\{x_{ij}\}, j = 0, \cdots, n$, where n is the number of the singular points. For $x_{i_0j_0}$ which is an evaluation point of $\tilde{\phi}''$, set $p \in S$ to be the minimal projection takes 1 at $x_{i_0j_0}$, and 0 at all the other evaluated point. By the same argument as above for p, get that $\tilde{\phi}''$ and $\tilde{\psi}''$ has the same multiplicity on $x_{i_0j_0}$. Therefore we can pair all the eigenvalue maps up to a small toleration. So, there is a unitary u_2 in $(1-p)A(1-p)$ such that

$$\left\| \tilde{\phi}''(f) - u_2\tilde{\psi}''(f)u_2^* \right\| \leq \varepsilon, \quad \forall f \in \mathcal{F}.$$

Set $u = u_1 + u_2$. Then we get

$$\|\phi(f) - u\psi(f)u^*\| \leq \varepsilon \quad \forall f \in \mathcal{F}.$$

This proves the proposition. □

Thanks the stably uniqueness theorem of Lin in [18] which works in a very general context, we have a stably uniqueness theorem for TAS-algebras. First, let us introduce the following definition.

Definition 4.4. Let A be a unital C*-algebra and $u \in \mathrm{U}_0(A)$, the connected component of the unitary group of A containing the identity. Then the *exponential length of* u, denoted by $\mathrm{cel}(u)$, is

$$\inf\{\sum_{k=1}^{n} ||h_k|| : n \in \mathbb{N}, h_1, \cdots, h_n \in A^{s.a.},\ u = \exp(ih_1)\cdots\exp(ih_n)\}.$$

Duplicate the argument of Lin in [18], we have the following stably uniqueness theorem for TAS-algebras.

Theorem 4.5. *Let A be a unital simple C*-algebra which satisfies UCT, and let $\mathbf{L} : U(M_\infty(A)) \to \mathbb{R}^+$ be a map. Then for any finite subset $\mathcal{F}$ of A and $\varepsilon > 0$, there exist a finite subset $\mathcal{P} \subset P(A)$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$ and an integer n such that for any TAS-algebra B and any $\mathcal{G} - \delta$ multiplicative completely positive linear contractions $\phi, \psi, \sigma : A \to B$ with*

(1) $[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}$,
(2) *cel*$(\phi(u)^*\psi(u)) \leq \mathbf{L}(u)$ *for any* $u \in U(A) \cap \mathcal{P}$

and σ is unital, there is an unitary $u \in M_{n+1}(B)$ such that:

$$\|u^* diag(\phi(a), \sigma(a), \cdots, \sigma(a))u - diag(\psi(a), \sigma(a), \cdots, \sigma(a))\| \leq \varepsilon \quad \forall a \in \mathcal{F}.$$

In the case A is a simple separable TASI-algebra, we get a natural number n by applying the theorem above for a finite subset $\mathcal{F}$ and $\delta > 0$. By Theorem 4.1, there is a set of mutually orthogonal projections $q, p_1, \cdots, p_n$ with $q + p_1 + \cdots + p_n = 1$, $q \preceq p_1$ and $p_i \sim p_1$, a splitting interval sub-algebra S with $1_S = p_1$ and two $\mathcal{F}-\varepsilon$ multiplicative linear unital maps $\phi_0 : A \to qAq$, $\phi_1 : A \to S$, such that:

$$||x - \phi_0(x) \oplus (\underbrace{\phi_1(x) \oplus \cdots \oplus \phi_1(x)}_{n \text{ copies}})|| \leq \varepsilon$$

for any x in $\mathcal{F}$. With this n and the splitting interval algebra S, we may use Theorem 4.3 and Theorem 4.5 to get an unitary in B instead of the matrix algebra over B.

Theorem 4.6. *Let A be a simple separable TASI-algebra satisfying UCT, and let $\mathbf{L} : U(M_\infty(A)) \to \mathbb{R}^+$ be a map. Then for any finite subset $\mathcal{F}$ of A, and $\varepsilon > 0$, there exist a finite subset $\mathcal{P} \subset P(A)$, a finite subset $S \subset A$, $\delta_1 > 0$ and a natural number n such that there exist mutually orthogonal projections $q, p_1, \cdots, p_n$ with $q \preceq p_1$ and $p_1, \cdots, p_n$ mutually unitary equivalent, a C*-subalgebra*

S which is a splitting interval algebra with $1_S = p_1$ and unital $S - \delta_1/2$-multiplicative completely positive contractions $\phi_0 : A \to qAq$ and $\phi_1 : A \to S$ such that

$$||x - \phi_0(x) \oplus (\underbrace{\phi_1(x) \oplus \cdots \oplus \phi_1(x)}_{n \quad copies})|| \leq \delta_1/2$$

for all $x \in S$.

Moreover, there exist a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{P}_0$ of projections in $M_\infty(S)$, a finite subset $H \subset A_{s.a.}$, $\delta > 0$ and $\sigma > 0$ such that for any simple TAS-algebra B and two $S \cup \mathcal{G} - \delta$ multiplicative completely positive contractions $L_1, L_2 : A \to B$ for which ($\delta = \min \delta_0, \delta_1$)

(1) $[L_1]|_{\mathcal{P}\cup\mathcal{P}_0} = [L_2]|_{\mathcal{P}\cup\mathcal{P}_0}$,
(2) $||\tau \circ L_1(g) - \tau \circ L_2(g)|| < \delta$, *for all $g \in H$ and $\tau \in T(A)$,*
(3) $e = L_1 \circ \phi_0(1_A) = L_2 \circ \phi_0(1_A)$ *is a projection,*
(4) *cel*$((L_1 \circ \phi_0(u)^*)(L_2 \circ \phi_0(u))) \leq \mathbf{L}(u)$ *(in $U(eBe)$) for any $u \in U(A) \cap \mathcal{P}$,*

there is an unitary $u \in B$ such that:

$$\|u^* L_1(a) u - L_2(a)\| \leq \varepsilon \quad \forall a \in \mathcal{F}.$$

Proof. This theorem follows from Theorem 4.3, Theorem 4.1 and Theorem 4.5. □

5. A Classification Theorem

In this section, we are going to use the existence theorem and uniqueness theorem in the previous sections to prove a classification theorem for TASI-algebras by approximately intertwining arguments.

However, in order to apply the uniqueness theorem, the two approximate homomorphisms are required not only to induce the same map on the level of the invariant, but also to satisfy the condition of the boundedness of the exponential length. Note that for the C*-algebras in the class $\mathcal{CS}$, we use circle algebras to realize the torsion free part of the K_1-group. Moreover, the unitary groups of TAS-algebras share many of the properties of TAI-algebras. So one can use the same method as in [18] to control the exponential length. As in [18], we use the following notation.

For a unital C*-algebra A, let $\mathrm{CU}(A)$ denote the closure of the commutator subgroup of $\mathrm{U}(A)$. It is a normal subgroup of $\mathrm{U}(A)$, and $\mathrm{U}(A)/\mathrm{CU}(A)$ is commutative. If $\mathrm{K}_1(A) = \mathrm{U}(A)/\mathrm{U}_0(A)$ where $\mathrm{U}_0(A)$ is the component of the unitary group containing the identity, $\mathrm{CU}(A)$ will be inside $\mathrm{U}_0(A)$. For $u \in \mathrm{U}(A)$, we use $\bar{u}$ to denote the image of u in $\mathrm{U}(A)/\mathrm{CU}(A)$.

Lemma 5.1. *Let u be a unitary in a splitting interval algebra S with generic dimension n. For any $\varepsilon > 0$, there are continuous functions $s_i : [0,1] \to \mathbb{T}$, i=1,...,n, and unitaries $W \in S$ such that*

$$\|u - W^* \mathrm{diag}\{s_1, ..., s_n\} W\| \leq \varepsilon.$$

Proof. Denote by n the generic size of S, and denote by the partition of n at the points 0 and 1 by $(m_0^{(1)}, ..., m_0^{(l_0)})$ and $(m_1^{(1)}, ..., m_1^{(l_0)})$ respectively.

Let

$$W_0 = \mathrm{diag}\{W_{0,1}, ..., W_{0,l_0}\}$$

and

$$W_1 = \mathrm{diag}\{W_{0,1}, ..., W_{0,l_0}\}$$

be two unitary matrices with the the same diagonal block size as that of S at the endpoint 0 and 1 respectively, such that $W_0^* u(0) W_0$ and $W_1^* u(1) W_1$ are diagonal matrices.

Let W' be a unitary in S with $W'(0) = W_0$ and $W'(1) = W_1$.

Consider the unitary $v := (W')^* u W'$. It is clear that $v(0)$ and $v(1)$ are diagonal. Without loss of generality, we may assume that there exists $0 < \delta < 1/4$ such that the restrictions of v to $[0, \delta]$ and $[1 - \delta, 1]$ are constant.

Consider the restrictions of v to $[\delta, 1-\delta]$, it is well-known that there is a unitary $V \in \mathrm{C}([\delta, 1-\delta], \mathrm{M}_n(\mathbb{C}))$ such that

$$\|V^*(t)v(t)V(t) - \mathrm{diag}\{s_1(t), ..., s_n(t)\}\| < \varepsilon \quad \text{forany } t \in [\delta, 1-\delta]$$

for some continuous functions $s_i : [\delta, 1-\delta] \to \mathbb{T}$. Moreover, the unitary V can be chosen such that

$$V^*(\delta)v(\delta)V(\delta) = v(\delta) \quad \text{and} \quad V^*(1-\delta)v(1-\delta)V(1-\delta) = v(1-\delta).$$

Let w_0 and w_1 be one branch of natural logarithm of $V(\delta)$ and $V(1-\delta)$ respectively such that they are well-defined. Extend the unitary V to $[0, 1]$ by $V(t) = \exp(\frac{t}{\delta} w_0)$ for any $t \in [0, \delta]$ and $V(t) = \exp(\frac{t-1+\delta}{\delta} w_0)$ for any $t \in [1-\delta, 1]$. Since $V(\delta)$ and $V(1-\delta)$ commute with $v(\delta)$ and $v(1-\delta)$ respectively, one has that w_0 and w_1 commute with $v(\delta)$ and $v(1-\delta)$ respectively, and hence $V(t), t \in [0, \delta]$ and $V(t), t \in [1-\delta, 1]$ commute with $v(\delta)$ and $v(1-\delta)$ respectively.

Extend each function $s_i(t)$ to the interval $[0, 1]$ such that the restrictions of s_i to $[0, \delta]$ and $[\delta, 1]$ are constant. Since that $V(0)$ and $V(1)$ are diagonal, one has that $V \in S$. Then, a direct calculation shows that

$$\|V^*(t)v(t)V(t) - \mathrm{diag}\{s_1(t), ..., s_n(t)\}\| \le \varepsilon \quad \text{for any } t \in [0, 1].$$

Therefore, set $W := W'V$, one has that

$$\|W^* u W - \mathrm{diag}\{s_1, ..., s_n\}\| \le \varepsilon,$$

as desired. □

Corollary 5.2. *Let u be a unitary in a splitting interval algebra S. For any $\varepsilon > 0$, there is an sel-adjoint element $h \in S$ such that*

$$\|u - \exp(ih)\| \le \varepsilon.$$

Proof. Since for any continuous function $s : [0, 1] \to \mathbb{T}$, there is a function $h : [0, 1] \to \mathbb{R}$ such that $s = \exp(ih)$, the corollary follows from Lemma 5.1. □

Lemma 5.3. *Let u be a unitary in a splitting interval algebra S. If $\det_t(u) = 1$ for any $t \in \mathrm{Sp}(S)$, then $u \in \mathrm{CU}(S)$.*

Proof. Denote by n the generic size of S, and denote by the partition of n at the points 0 and 1 by $(m_0^{(1)}, ..., m_0^{(l_0)})$ and $(m_1^{(1)}, ..., m_1^{(l_0)})$ respectively.

By Lemma 5.1, we may assume that $u = \mathrm{diag}\{s_1(t), ..., s_n(t)\} \in S$. Since $\det_t(u) = 1$ for any $t \in \mathrm{Sp}(S)$, one has that

$$\prod_{i \in m_0^{(k)}} s_i(0) = 1 \quad \text{and} \quad \prod_{i \in m_1^{(k)}} s_i(1) = 1 \tag{5.1}$$

for each k.

Since $\mathrm{CU}(S)$ is closed, without loss of generality, let us assume that each function $s_i(t)$ is constant if $t \in [0, \delta]$ and $t \in [1-\delta, 1]$ for some $\delta > 0$. Denote by

$$u_1 = \mathrm{diag}(1, s_1, s_2 s_1, ..., s_{n-1}s_{n-2}\cdots s_1) \in S,$$

and consider

$$uu_1 = \mathrm{diag}(s_1, s_2s_1, s_3s_2s_1, ..., s_ns_{n-1}\cdots s_1).$$

Using 5.1, one can find a unitaries $W_0 = \mathrm{diag}(W_{1,0}, ..., W_{l_0,0})$ and $W_1 = \mathrm{diag}(W_{1,1}, ..., W_{l_1,1})$ such that each $W_{i,j}$ is an $|m_j^{(i)}| \times |m_j^{(i)}|$ matrix, and

$$uu_1(0) = W_0^* u_1(0) W_0 \quad \text{and} \quad uu_1(1) = W_1^* u_1(1) W_1.$$

Moreover, there is a $n \times n$ matrix V such that

$$uu_1(t) = V^* u_1(t) V \quad \text{for any } t \in [\delta, 1-\delta].$$

Consider the path of unitaries

$$W_0(t) = W_0 \exp(t \ln(W_0^* V)) \quad \text{and} \quad W_1(t) = V \exp(t \ln(V^* W_1)).$$

Then, $W_0(0) = W_0$ and $W_0(1) = V$. Note that

$$V^* W_0 uu_1(0) W_0^* V = V^* u_1(0) V = V^* u_1(\delta) V = uu_1(0).$$

We have that for any $t \in [0, 1]$,

$$\exp(t \ln(V^* W_0)) uu_1(0) \exp(t \ln(W_0^* V)) = uu_1(0),$$

and hence

$$\begin{aligned} W_0^*(t) u_1(0) W_0(t) &= \exp(t \ln(V^* W_0)) W_0^* u_1(0) W_0 \exp(t \ln(W_0^* V)) \\ &= \exp(t \ln(V^* W_0)) uu_1(0) \exp(t \ln(W_0^* V)) \\ &= uu_1(0). \end{aligned}$$

With the same argument, we have a path of unitaries $W_1(t)$ such that $W_1(0) = V$, $W_1(1) = W_1$, and

$$W_1^*(t) u_1(1) W_1(t) = uu_1(1).$$

Denote by

$$W(t) = \begin{cases} W_0(\frac{t}{\delta}) & \text{if } t \in [0, \delta] \\ V & \text{if } t \in [\delta, 1-\delta] \\ W_1(\frac{1-\delta-t}{\delta}) & \text{if } t \in [1-\delta, 1] \end{cases}$$

It is clear that $W(t) \in S$ and $uu_1 = W^* u_1 W$. Hence one has that $u = u_1^* W^* u W \in \mathrm{CU}(S)$. □

Lemma 5.4. *For any $\varepsilon > 0$, there is a constant K such that for any splitting interval algebra S and a unitary $u \in S$, if for any irreducible representation π_t of S, $\dim(\pi_t) > K$ and $\det(\pi_t(u)) = 1$, then*

$$\|u - \exp(ih_1)\exp(ih_2)\exp(ih_3)\| \leq \varepsilon$$

for some self-adjoint element h_i with $\|h_i\| \leq 2\pi$, $i = 1, ..., 3$.

Proof. Without loss of generality, one may assume that there exists $0 < \delta < 1/4$ such that the restrictions of u to $[0, \delta]$ and $1 - \delta, 1$ are constant. Denote by the constant matrices by u_0 and u_1, and assume that u and v can be diagonalized by unitary matrices W_0' and W_1'. Noting that W_0' and W_1 have the same block diagonal form as u in 0 and 1, there are unitaries $W_0 \in S$ and $W_1 \in S$ such that $W_0(t) = W_0'$ and $W_1(t) = I$ for $t \in [0, \delta]$, and $W_0(t) = I$ and $W_1(t) = W_1'$ for $t \in [1 - \delta, 1]$. Thus, by consider $(W_0W_1)^*u(W_0W_1)$, we may assume that $u(t)$ is diagonal for $t \in [0, \delta] \cup [1 - \delta, 1]$.

Denote by K the constant of Theorem 3.3 of [24] corresponding to $X = [\delta, 1 - \delta]$ and ε, and consider the restriction of u to the interval $[\delta, 1 - \delta]$. It follows from Theorem 3.3 of [24] that there exist self-adjoint functions $h_1, h_2, h_3 : [\delta, 1 - \delta] \to \mathrm{M}_n(\mathbb{C})$ such that

$$\|u(t) - \exp(ih_1(t))\exp(ih_2(t))\exp(ih_3(t))\| \leq \varepsilon, \quad \forall t \in [\delta, 1 - \delta],$$

and $\|h_i\| \leq 2\pi$, i=1, 2, 3. In order to proof the lemma, one has to show that for each h_i, the matrices $h_i(\delta)$ and $h_i(1 - \delta)$ has the right diagonal form to fit into the splitting points of S.

By checking the proof of Theorem 3.3 of [24], one has that the matrices $h_1(\delta)$ and $h_1(1 - \delta)$ are diagonal (since the unitary u is diagonal at the points δ and $1 - \delta$, the unitary u_4 in the step 3 of the proof can be chose to be diagonal at these two points, and hence in the step 4 of the proof, $h_1(x) = g^{(x)}(u_4(x))$ is diagonal if $x = \delta$ or $x = 1 - \delta$).

By checking the step 5 and the step 6 of proof of Theorem 3.3 of [24], one has that the unitary v_5 and v_6 can be chose in such a way that their restrictions to δ and $1 - \delta$ are inside one of the diagonal blocks (which has size at least K). Hence there is a projection p in $\mathrm{M}(\mathrm{C}([\delta, 1 - \delta]))$ such that p has rank at least K, $p(\delta)$ and $p(1 - \delta)$ are inside the corresponding diagonal blocks, and v_5 and v_6 are in the hereditary sub-C*-algebra generated by p.

Consider the unital hereditary sub-C*-algebra generated by p and the element $v_6 \oplus 1_K$ inside this sub-C*-algebra, and applying step 7 of the proof of Theorem 3.3 of [24]. There are elements h_2 and h_3 such that

$$\|v_6 \oplus (p - v_6^*v_6) - \exp(ih_2')\exp(ih_3')\| < 2\varepsilon/5.$$

Since h_2' and h_3' are in the hereditary sub-C*-algebra generated by p, the elements $h_2 = h_2' \oplus (1-p)$ and $h_3 = h_3' \oplus (1 - p)$ has the right form of diagonal blocks at δ and $1 - \delta$, and

$$\|u_6 - \exp(ih_2)\exp(ih_3)\| < 2\varepsilon/5.$$

Therefore,

$$\|u(t) - \exp(ih_1(t))\exp(ih_2(t))\exp(ih_3(t))\| \leq \varepsilon, \quad \forall t \in [\delta, 1 - \delta],$$

and the matrices $h_i(\delta)$ and $h_i(1 - \delta)$ has the right diagonal form to fit into the splitting points of S respectively. Extend h_i to the whole interval $[0, 1]$ constantly on $[0, \delta]$ and $[1 - \delta, 1]$. Then, each h_i induces an element of S. Using the assumption that u is constant on $[0, \delta]$ and $[\delta, 1]$, one has that

$$\|u(t) - \exp(ih_1(t))\exp(ih_2(t))\exp(ih_3(t))\| \leq \varepsilon, \quad \forall t \in [0, 1],$$

as desired. □

If $\phi : A \to B$ is a unital *-homomorphism, it will induce a homomorphism $\phi^\ddagger : \mathrm{U}(A)/\mathrm{CU}(A) \to \mathrm{U}(B)/\mathrm{CU}(B)$. Moreover, for any finite subset $\mathcal{U} \subset \mathrm{U}(A)$ and any $\varepsilon > 0$, denote by $\bar{F}$ the subgroup of $\mathrm{U}(A)/\mathrm{CU}(A)$ generated by $\mathcal{U}$. We can choose a finite subset $\mathcal{G} \subset A$ and $\delta > 0$, such that for any $L : A \to B$ which is a $\mathcal{G} - \delta$-multiplicative completely positive linear contraction, there is a homomorphism $L^\ddagger : \bar{F} \to \mathrm{U}(B)/\mathrm{CU}(B)$ with $||\overline{L(u)} - L^\ddagger(\bar{u})|| < \varepsilon$ for any $u \in \mathcal{U}$. We say $L^\ddagger$ is induced by L.

Theorem 5.5 (See Theorem 6.5 of [18]). *Let A be a simple TAS-algebra, and let $u \in \mathrm{U}_0(A)$. Then, for any $\varepsilon > 0$, there are unitaries u_1 and u_2 such that u_1 has exponential length no more than 2π, u_2 is an exponential, and*

$$\|u - u_1u_2\| < \varepsilon.$$

Moreover, $\mathrm{cer}(A) \leq 3 + \varepsilon$.

Proof. Using the fact that the exponential rank of splitting interval algebra is $1 + \varepsilon$ (Corollary 5.2), one can repeat the proof of Theorem 6.5 of [18]. □

Lemma 5.6. *Let A be a simple TAS-algebra, and let $u \in \mathrm{CU}(A)$. Then, $u \in \mathrm{U}_0(A)$ and* $\mathrm{cel}(u) \leq 8\pi$.

Proof. The proof is a repeating of that of Lemma 6.9 of [18]. Instead appealing to 3.4 of [25], one uses Lemma 5.4 in the proof. □

Theorem 5.7 (See Theorem 6.10 of [18]). *Let A be a simple TAS-algebra. Let $u, v \in \mathrm{U}(A)$ such that $[u] = [v]$ in $K_1(A)$ and*

$$u^k, v^k \in \mathrm{U}_0(A) \quad \textit{and} \quad \mathrm{cel}((u^*)^k u^k) \leq L.$$

Then,

$$\mathrm{cel}(u^*v) \leq 8\pi + L/k.$$

Moreover, there is $y \in \mathrm{U}_0(A)$ with $\mathrm{cel}(y) \leq L/k$ such that $\overline{u^*v} = \overline{y}$ in $\mathrm{U}(A)/\mathrm{CU}(A)$.

Proof. The proof is similar to that of Theorem 6.10 of [18]. Write

$$u^*v = \prod_j \exp(ia_i) \quad \text{and} \quad (u^k)^*v^k = \prod_m \exp(ib_m),$$

where a_j and b_m are self-adjoint. We may assume that $\sum \|b_m\| < L$, since $\mathrm{cel}((u^*)^k u^k) \leq L$. Write $M = \sum \|b_m\|$. Since A is a TAS-algebra, for any $\delta > 0$ with $\delta/(1-\delta) < \varepsilon/2(M + L + 1)$ and sufficiently small $\eta > 0$ and sufficiently large finite subset $\mathcal{G}$, there is a projection $p \in A$ and a sub-C*-algebra $S \subset A$ with $1_S = p$ such that S is a direct sum of splitting interval algebras, and

(1) $pxp \in_\eta S$ for any $x \in \mathcal{G}$,

(2) $\|u - u_0 \oplus u_1\| \leq \eta$ and $\|v - v_0 \oplus v_1\| \leq \eta$ for some unitaries u_0, v_0 in $(1-p)A(1-p)$ and u_1, v_1 in pAp,
(3) $\mathrm{cel}(u_0^* v_0) \leq M+1$ in $(1-p)A(1-p)$ and $\mathrm{cel}((u_1^k)^* v_1^k) < L$ in S,
(4) $\tau(1-p) \leq \delta$ for any $\tau \in \mathrm{T}(A)$.

Without loss of generality, we may assume that the dimension of any nonzero irreducible representation of S is greater that $M := \max(2\pi^2/\varepsilon, K)$, where K is the constant of Lemma 5.4.

Let us assume that S is consisted of one direct summand and has generic dimension n. Without loss of generality, we may assume that there exists $0 < \delta < 1/4$ such that $(u_1^*)^k v_1^k$ is constant on $[0, \delta]$ and on $[1-\delta, 1]$. Consider the restriction of S to $[\delta, 1-\delta]$. By Lemma 3.3 (1) of [25], there exists $a \in (\mathrm{M}_n(\mathrm{C}([\delta, 1-\delta])))_{\mathrm{s.a.}}$ such that

$$\det(\exp(ia)(u_1^*)^k v_1^k) = 1 \quad \text{for any } t \in [\delta, 1-\delta].$$

Using the connectivity of the unitary subgroup of a matrix algebras, one may assume further that

$$a(\delta) = \mathrm{diag}\{\underbrace{r_1, ..., r_{m_0}}_{m_0^1}, \underbrace{r_{m_0^{(1)}}, ..., r_{m_0^{(1)}+m_0^{(2)}}}_{m_0^1}, ..., \underbrace{r_{n-m_0^{(l_n)}}, ..., r_n}_{m_0^{l_0}}\}$$

and

$$a(1-\delta) = \mathrm{diag}\{\underbrace{s_1, ..., s_{m_0}}_{m_1^1}, \underbrace{s_{m_1^{(1)}}, ..., s_{m_1^{(1)}+m_1^{(2)}}}_{m_1^1}, ..., \underbrace{s_{n-m_1^{(l_n)}}, ..., s_n}_{m_1^{l_1}}\}$$

with products of each group of r's or s's equal to one. Therefore, one can extend a constantly to $[0, 1]$ to get an element in S. Since the restriction of $(u_1^*)^k v_1^k$ is constant on $[0, \delta]$ and on $[1-\delta, 1]$, one has that

$$\det(\exp(ia)(u_1^*)^k v_1^k) = 1 \quad \text{for any } t \in \mathrm{Sp}(S).$$

Therefore

$$\det((\exp(ia/k)u_1^* v_1)^k) = 1 \quad \text{for any } t \in \mathrm{Sp}(S),$$

and

$$\det(\exp(ia/k)u_1^* v_1) = \exp(i2l\pi/k) \quad \text{for any } t \in (0, 1),$$

for some $l \in 0, 1, ..., k-1$. Define the function f in the following way: On $[\delta, 1-\delta]$, set $f = -2l\pi/(nk)1_n$; on points 0 and 1, define $a_i = -\sum_{j=1}^{m_0^{(i)}} r_j/m_0^{(i)}$ and $b_i = -\sum_{j=1}^{m_1^{(i)}} r_j/m_1^{(i)}$

Set

$$f(0) = \mathrm{diag}\{\underbrace{a_1, ..., a_1}_{m_0^1}, \underbrace{a_2, ..., a_2}_{m_0^1}, ..., \underbrace{a_{l_0}, ..., a_{l_0}}_{m_0^{l_0}}\},$$

and

$$f(1) = \mathrm{diag}\{\underbrace{b_1, ..., b_1}_{m_1^1}, \underbrace{b_2, ..., b_2}_{m_1^1}, ..., \underbrace{b_{l_1}, ..., b_{l_1}}_{m_1^{l_1}}\},$$

and define

$$f(t) = (1 - \frac{t}{\delta})f(0) + \frac{t}{\delta}f(\delta), \quad t \in [0, \delta],$$

$$f(t) = (\frac{1-t}{\delta})f(\delta) + (1 + \frac{t-1}{\delta})f(1), \quad t \in [1-\delta, 1].$$

We then have $f \in S$ with $\|f\| \le \|a\|/M + 2\pi \le L/M + 2\pi$ and

$$\det(\exp(if)\exp(ia/k)u_1^*v_1) = 1 \quad \text{for any } t \in \mathrm{Sp}(S).$$

By Lemma 5.3, $z_1 := \exp(if)\exp(ia/k)u_1^*v_1 \in \mathrm{CU}(S)$. Moreover, by Lemma 5.4, there are unitaries w_1, w_2, and w_3 with $\|w_i\| \le 2\pi$ such that

$$\|\exp(if)\exp(ia/k)u_1^*v_1 - \exp(iw_1)\exp(iw_2)\exp(iw_3)\| \le \varepsilon,$$

and hence

$$\|u_1^*v_1 - \exp(if)\exp(ia/k)\exp(iw_1)\exp(iw_2)\exp(iw_3)\| \le \varepsilon.$$

Therefore,

$$\mathrm{cel}(u_1^*v_1) \le L/M + L/k + 6\pi.$$

Then, by Lemma 6.4 of [18], there is $y' \in \mathrm{CU}(A)$ and $y'' \in \mathrm{U}_0(A)$ such that $(u_0 \oplus p)^*(v_0 \oplus p) = y'y''$ and $\mathrm{cel}(\mathrm{y}'') < \varepsilon/2$. Note that $((1-p) \oplus z_1)y' \in \mathrm{CU}(A)$. Therefore

$$\overline{u^*v} = \overline{\exp(if)\exp(ia)w}$$

for some $w_0 \in \mathrm{U}_0(A)$ with $\mathrm{cel}(w) \le \varepsilon/2$ for sufficiently small η. By Lemma 5.6, one has that

$$\mathrm{cel}(u^*v) \le 2\pi/M + L/k + 8\pi + \varepsilon/2 < 8\pi + L/k + \varepsilon,$$

as desired. ☐

Theorem 5.8 (See Theorem 6.11 of [18]). *Let A be a simple TAS-algebra. Then the the group $\mathrm{U}_0(A)/\mathrm{CU}(A)$ is torsion free.*

Proof. The proof is the same as that of Theorem 6.11 of [18], and also essentially the same as that of Theorem 5.7. ☐

Corollary 5.9 (See Corollary 6.12 of [18]). *Let B_n be a sequence of unit simple TAS-algebras. Let $\prod_n^b K_1(B_n)$ be the set of sequence $z = \{z_n\}$, where $z_n \in K_1(B_n)$ and each z_n can be represented by a unitary in a matrix algebra over B_n. Then, the kernel of the map*

$$K_1(\prod_n B_n) \to \prod_n K_1(B_n) \to 0$$

is a divisible and torsion free subgroup of $K_1(\prod_n B_n)$.

Proof. Using Theorem 5.5 and Theorem 5.7 instead of 6.5 and 6.10 of [18], one can repeat the argument of 6.12 of [18]. ☐

Let C' be a homogeneous C*-algebra $P\mathrm{M}_n(\mathrm{C}(X))P$, where $X = S^1 \vee \cdots \vee S^1 \vee Y$ for some finite CW complex Y with torsion K_1 and dimension no more than 3, and P is a projection in $\mathrm{M}_n(\mathrm{C}(X))$ with rank $r \ge 6$. Then $\mathrm{K}_1(P\mathrm{M}_n(\mathrm{C}(X))P) = \mathrm{Tor}(\mathrm{K}_1(C')) \oplus G_1$ where G_0 is s copies of $\mathbb{Z}$ and G_1 is the torsion part of $\mathrm{K}_1(C)$. Denote by $D' = \bigoplus_{i=1}^s E_i$, where $E_i \cong \mathrm{M}_r(\mathrm{C}(\mathbb{T}))$. Then, there is an obvious map $\Pi : P\mathrm{M}_n(\mathrm{C}(X))P \to D'$. We have that $\mathrm{K}_1(D') \cong G_1$ and it is not surjective if $s \ge 2$. Denote by $\Pi_i : P\mathrm{M}_n(\mathrm{C}(X))P \to E_i$ to be the composition of Π with the projection from D' to E_i.

Denote by C a finite direct sum of the C*-algebras of form C', matrix algebras, splitting interval algebras, and $p\mathrm{M}_m(\mathrm{C}(Y))p$ with $\dim(Y) \leq 3$, $\mathrm{Rank}(p) \geq 6$, and $\mathrm{K}^1(Y)$ is finite. Write D by the direct sum of D's in the C*-algebras of form C'. Then, It follows from 7.1 of [18] that

$$\mathrm{U}(C)/\mathrm{CU}(C) = \mathrm{U}_0(C)/\mathrm{CU}(C) \oplus G_1 \oplus \mathrm{Tor}(\mathrm{K}_1(C)).$$

We will set π_0, π_1, π_2 to be the projection maps from $\mathrm{U}(C)/\mathrm{CU}(C)$ to each component according to the decomposition above. As in [18], we have the following lemmas to control the exponential length in the classification theorem. The proofs are repeating the arguments in [18].

Lemma 5.10 (See Lemma 7.2 of [18]). *Let $C = \oplus_{i=1}^{l+l_1} C_i$ be as above and $\mathcal{U} \subset \mathrm{U}(C)$ be a finite subset and F be the group generated by $\mathcal{U}$. Suppose that G is a subgroup of $\mathrm{U}(C)/\mathrm{CU}(C)$ which contains $\bar{F}$, $\pi_1(\mathrm{U}(C)/\mathrm{CU}(C))$, and $\pi_2(\mathrm{U}(C)/\mathrm{CU}(C))$. Suppose that the composition map $\gamma : \bar{F} \to \mathrm{U}(D)/\mathrm{CU}(D) \to \mathrm{U}(D)/\mathrm{U}_0(D)$ is injective and $\gamma(\bar{F})$ is free. Let B be a unital C*-algebra and $\Lambda : G \to \mathrm{U}(B)/\mathrm{CU}(B)$ be a homomorphism such that $\Lambda(G \cap \mathrm{U}_0(C))/\mathrm{CU}(C) \subset \mathrm{U}_0(B)/\mathrm{CU}(B)$. Then there are homomorphism $\beta : \mathrm{U}(D)/\mathrm{CU}(D) \to \mathrm{U}(B)/\mathrm{CU}(B)$ with $\beta(\mathrm{U}_0(D)/\mathrm{CU}(D)) \subset \mathrm{U}(B)/\mathrm{CU}(B)$, and $\theta : \pi_2(\mathrm{U}(C)/\mathrm{CU}(C)) \to \mathrm{U}(B)/\mathrm{CU}(B)$ such that*

$$\beta \circ \Pi^{\ddagger} \circ \pi_1(\bar{w}) = \Lambda(\bar{w})(\theta \circ \pi_2(\bar{w_2}))$$

for any $w \in F$ and such that $\theta(g) = \Lambda\_\pi_2(\mathrm{U}(C)/\mathrm{CU}(C))(g^{-1})$ for any $g \in \pi_2(\mathrm{U}(C)/\mathrm{CU}(C))$. Moreover, $\beta(\mathrm{U}_0(D)/\mathrm{CU}(D)) \subset \mathrm{U}_0(B)/\mathrm{CU}(B)$.

If furthermore B is a simple TAS-algebra and $\Lambda(\mathrm{U}(C)/\mathrm{CU}(C)) \subset \mathrm{U}_0(B)/\mathrm{CU}(B)$, then $\beta \circ \Pi^{\ddagger} \circ (\pi_1)|_{\bar{F}} = \Lambda|_{\bar{F}}$.

Proof. The first part of the statement is exactly the same as that of Lemma 7.2 of [18]. Noting that $\mathrm{U}_0(B)/\mathrm{CU}(B)$ is torsion free by Theorem 5.8, then the second part of the statement follows. □

Lemma 5.11 (See Lemma 7.3 of [18]). *Let B be a separable simple TAS-algebra and let C be as above. Let $\mathcal{U} \subset U(B)$ be a finite subset and let F be the subgroup generated by $\mathcal{U}$ such that $\kappa_1(\bar{F})$ is free, where $\kappa_1 : U(B)/CU(B) \to K_1(B)$ is the quotient map. Suppose that $\alpha : K_1(C) \to K_1(B)$ is a one-to-one homomorphism and $L : \bar{F} \to U(C)/CU(C)$ is a ono-to-one homomorphism with $L(\bar{F} \cap U_0(C)/CU(C)) \subset U_0(B)/CU(B)$ such that $\pi_1 \circ L$ is one-to-one and*

$$\alpha \circ \kappa_1' \circ L(g) = \kappa_1(g) \quad \text{for all} \quad g \in \bar{F},$$

where $\kappa_1' : U(C)/CU(C) \to K_1(C)$ is the quotient map. Then there exists a homomorphism $\beta : U(C)/CU(C) \to U(B)/CU(B)$ with $\beta(U_0(C)/CU(C)) \subset U_0(B)/CU(B)$ such that

$$\beta \circ L(f) = f$$

for all $f \in \bar{F}$.

Proof. The proof is exactly a repeating the that of Lemma 7.3 of [18]. □

Lemma 5.12 (See Lemma 7.4 of [18]). *Let B be a simple separable TAS-algebra and C as above. Let F be a group generated by a finite subset $\mathcal{U} \subset U(C)$ such that $(\pi_1)|_{\bar{F}}$ is one-to-one. Let G be a subgroup containing $\bar{F}$, $\pi_1(U(C)/CU(C))$ and $\pi_2(U(C)/CU(C))$. Suppose that*

$\alpha : U(C)/CU(C) \to U(B)/CU(B)$ is a homomorphism ($\alpha(U_0(C)/CU(C)) \subset U_0(B)/CU(B)$). Then for any $\varepsilon > 0$, there is $\delta > 0$ satisfying the following: if $\phi = \phi_0 \oplus \phi_1 : C \to B$ is a $\mathcal{G} - \eta$-multiplicative completely positive linear contraction such that

(1) *both ϕ_0 and ϕ_1 are $\mathcal{G} - \eta$-multiplicative,*
(2) *$\mathcal{G}$ is sufficiently large and η is sufficiently small depending only on F and C (such that $\phi^\ddagger$ is well defined on a subgroup of $U(C)/CU(C)$ containing all of $\bar{F}$, $\pi_0(\bar{F})$, $\pi_1(U(C)/CU(C))$, and $\pi_2(U(C)/CU(C))$),*
(3) *ϕ_0 is homotopically trivial (homotopic to a point evaluation), $(\phi_0)_{*0}$ is well-defined and $[\phi]|_{K_1(C)} = \alpha_*$,*
(4) *$\tau(\phi_0(1_C)) < \delta$ for all $\tau \in T(B)$ (assume $e_0 = \phi_0(1_C)$),*

then there is a homomorphism $\Phi : C \to e_0Be_0$ such that

(1) *Φ is homotopically trivial and $(\Phi)_{*0} = (\phi)_{*0}$ and*
(2) *$\alpha(\bar{w})^{-1}(\Phi \oplus \phi_1)^\ddagger(\bar{w}) = \bar{g_w}$ where $g_w \in U_0(B)$ and $cel(g_w) < \varepsilon$ for any $w \in \mathcal{U}$.*

Proof. By Theorem 5.8, the group U(B)/CU(B) is torsion free. One then can repeat the argument of Lemma 7.4 of [18]. □

Lemma 5.13 (See Lemma 7.5 of [18])**.** *Let B be a separable simple TAS-algebra and C as above. Let $\mathcal{U} \subset U(B)$ be a finite subset and F be the subgroup generated by $\mathcal{U}$ such that $\kappa_1(\bar{F})$ is free, where $\kappa : U(B)/CU(B) \to K_1(B)$ is the quotient map. Let $\phi : C \to B$ be a homomorphism such that $(\phi)_{*1}$ is one-to-one. Suppose that $j, L : \bar{F} \to U(C)/CU(C)$ are two one-to-one homomorphisms with $j(\bar{F})$,$L(\bar{F}) \subset U_0/CU(C)$ such that $\kappa_1 \circ \phi^\ddagger \circ L = \kappa_1 \circ \phi^\ddagger \circ j = \kappa_1|_{\bar{F}}$, and they are one-to-one.*

Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\phi = \phi_0 \oplus \phi_1 : C \to B$, where ϕ_0 and ϕ_1 are homomorphisms satisfying the following:

(1) *$\tau(\phi_0(1_C)) < \delta$ for all $\tau \in T(B)$ and*
(2) *ϕ_0 is homotopically trivial,*

then there is a homomorphism $\psi : C \to e_0Be_0$ ($e_0 = \phi_0(1_C)$) such that

(1) *$[\psi] = [\phi_0]$ in $Hom_\Lambda(\underline{K}(C), \underline{K}(B))$ and*
(2) *$(\phi^\ddagger \circ j(\bar{w}))^{-1}(\psi \oplus \phi_1)^\ddagger(L(\bar{w})) = \bar{g_w}$ where $g_w \in U_0(B)$ and $cel(g_w) < \varepsilon$ for any $w \in \mathcal{U}$.*

Proof. The argument of Lemma 7.5 of [18] can be duplicated: In the proof, instead using Lemma 7.4 of [18], one uses Lemma 5.12. □

With those lemmas, we have the following classification theorem. The proof is also a duplicate of Lin's argument of the classification theorem for tracially AI-algebras in [18].

Theorem 5.14. *Let A be a simple separable TAS-algebra which satisfies UCT. If there is a simple inductive limit C*-algebra B as in Section 2.3 (the class $\mathcal{CS}$) such that*

$$(K_0(A), K_0(A)^+, [1_A]_0, K_1(A), T(A)) \cong (K_0(B), K_0(B)^+, [1_B]_0, K_1(B), T(B)),$$

then $A \cong B$. Moreover, the $*$-isomorphism can be chosen to induce the isomorphism on the invariants.

Proof. Denote by κ the isomorphism

$$\kappa : (\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]_0; \mathrm{K}_1(A)) \to (\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B]_0; \mathrm{K}_1(B)),$$

and let θ be the isomorphism from $T(B)$ to $T(A)$ compatible with κ. Since A and B satisfy the UCT, there is $\alpha \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))^+$ which induces κ. Moreover, α can be chosen to be invertible. Let $\mathbb{L} : \mathrm{U}(B) \to \mathbb{R}^+$ be defined as follows:

$$\mathbb{L}(u) = \begin{cases} 2\mathrm{cel}(u) + 8\pi + \pi/16 & \text{if } u \in \mathrm{U}_0(B) \\ 16\pi + \pi/16 & \text{if } u \notin \mathrm{U}_0(B) \text{ and } [u]_1 \text{ is torsion-free in } \mathrm{K}_1(B) \\ (2\mathrm{cel}(u^k))/k + 16\pi + \pi/16 & \text{if } u \notin \mathrm{U}_0(B) \text{ and } [u]_1 \text{ has torsion of order } k. \end{cases}$$

Fix $\varepsilon > 0$ and finite subset $\mathcal{F} \subset B$. Let $\delta' > 0$, the natural number n, finite subset $\mathcal{P} \subset P(B)$, finite subsets $S \subset B$ be as required in Theorem 4.6. Then there exist mutually orthogonal projections $q, p_1, \cdots, p_n$ with $q \preceq p_1$ and $p_1, \cdots, p_n$ mutually unitary equivalent, a C*-subalgebra S_1 which is a splitting interval algebra with $1_{S_1} = p_1$ and unital $S - \delta_1'/2$-multiplicative completely positive contractions $h_0 : B \to qBq$ with $h_0(x) = qxq$, and $h_1 : B \to S_1$ such that

$$||x - h_0(x) \oplus (\underbrace{h_1(x) \oplus \cdots \oplus h_1(x)}_{n \quad \text{copies}})|| \le \delta_1'/16$$

for all $x \in S$. Put $S = \mathrm{M}_n(S_1) \subset (1-q)B(1-q)$. Let $\mathcal{P}_0, \mathcal{G}_0, H, \delta_0$ and σ_1 be required by Theorem 4.6. Set $\delta = \min\{\delta_0, \delta'\}$. We may assume that $\mathcal{P}_0$ contains the minimal projections of S which present a generating set of the positive cone of $\mathrm{K}_0(S)$.

Without loss of generality, we may assume for each $u \in \mathrm{U}(B) \cap \mathcal{P}_0$ has the form $quq \oplus (1-q)u(1-q)$, where $quq \in \mathrm{U}(qBq)$ and $(1-q)u(1-q) \in \mathrm{U}(C)$. Since B is the inductive limit of B_i, we also assume $q \in B_1$ and $quq \in \mathrm{U}(qB_1q)$. Let $\mathcal{U}' = \{quq, u \in \mathrm{U}(B) \cap \mathcal{P}\}$ and let F be the subgroup of $\mathrm{U}(qBq)$ generated by $\mathcal{U}'$. Let $\bar{F}$ be the image of F in $\mathrm{U}(qBq)/\mathrm{CU}(qBq)$ where $\mathrm{CU}(qBq)$ is the commutator subgroup of $\mathrm{U}(qBq)$. By 6.6(3) of [18], we have $\bar{F} = (\bar{F} \cap \mathrm{U}_0(qBq)/\mathrm{CU}(qBq)) \oplus \bar{F}_0 \oplus \bar{F}_1$, where $\bar{F}_0$ is torsion and $\bar{F}_1$ is torsion free. Furthermore, we can assume $\mathcal{U}' = \mathcal{U}_0 \cup \mathcal{U}_1$ with $\bar{\mathcal{U}}_0$ generating $(\bar{F} \cap \mathrm{U}_0(qBq)/\mathrm{CU}(qBq)) \oplus \bar{F}_0$ and $\bar{\mathcal{U}}_1$ generating $\bar{F}_1$. We also assume $q \in B_1$ and $\mathcal{U}_0, \mathcal{U}_1 \subset qB_1q$. Note that $\mathrm{K}_1(B_m) \to \mathrm{K}_1(B_{m+1}) \to \mathrm{K}_1(B)$ is one-to-one for all m.

Let $\mathcal{G}_1$ be a finite subset of B containing $S, \mathcal{G}_0, H, \mathcal{U}', \{q, p_1, \cdots, p_n\}$ and a finite generating set of S. We assume $\mathcal{G}_1 \subset B_1$. By Theorem 3.24, there exists a $\mathcal{G}_1 - \delta/4$ multiplicative completely positive linear contraction $L_1 : B \to A$ such that

$$[L_1]|_{\mathcal{P} \cup \mathcal{P}_0} = \alpha^{-1}|_{\mathcal{P} \cup \mathcal{P}_0}$$

and

$$|\theta^{-1}(\tau)(a) - \tau(L_1(a))| \le \sigma/2 \text{ for all } a \in H, \tau \in T(A).$$

We assume that $L_1^{\ddagger}$ is well defined on $\bar{F}$. Define $\mathbb{L}_1(A) \to \mathbb{R}^+$ in the same manner as the $\mathbb{L}$. Let $\mathcal{F}_1$ be a finite subset of A. Let $\delta_1' > 0$, the natural number n_1, finite subset $\mathcal{P}_1 \subset P(A)$, finite subsets $\mathcal{S}_1 \subset A$ as required in Theorem 4.6 (for $A, \mathbb{L}_1, \mathcal{F}_1$ and $\varepsilon/4$). Then there exist mutually orthogonal projections $q', p_1', \cdots, p_n'$ with $q' \preceq p_1'$ and $p_1', \cdots, p_n'$ mutually unitary equivalent, a C*-subalgebra S_2' which is a splitting interval algebra with $1_{S_2'} = p_1'$ and unital

$\mathcal{S}_1 - \delta_1'/4$-multiplicative completely positive contractions $h_0' : A \to q'Aq'$ with $h_0'(x) = q'xq'$, and $h_1 : A \to S_1'$ such that

$$||x - h_0'(x) \oplus (\underbrace{h_1'(x) \oplus \cdots \oplus h_1'(x)}_{n_1 \text{ copies}})|| \leq \delta_1'/16$$

for all $x \in S_2'$. We assume $L_1(\mathcal{S}) \subset \mathcal{S}_1$. Set $S_2 = M_n(S_2') \subset (1-q')A(1-q')$, and let $\mathcal{P}_{01}, \mathcal{G}_{01}, H_1, \delta_{0,1}$ and σ_{01} also be as required by Theorem 4.6. Let $\delta_1 = \min\{\delta_1', \delta_{01}\}$. We may assume that $\delta_1 < \delta/2, \sigma_1 < \sigma/4$ and $\mathcal{P}_{01}$ contains the minimal projections of S_2 which present the generating set of the positive cone of $\mathrm{K}_0(S_1)$. We also assume that q' commutes with each elements of H_1 and $\mathcal{S}_1$, and $[L_1](\mathcal{P} \cup \mathcal{P}_0) \subset [\mathcal{P}_1]$.

Again, we assume for each $u \in \mathrm{U}(A) \cap \mathcal{P}_1$ has the form $q'uq' \oplus (1-q')u(1-q')$, where $q'uq' \in \mathrm{U}(q'Aq')$ and $(1-q')u(1-q') \in \mathrm{U}(S_1)$. Let $\mathcal{V}' = \{q'uq', u \in \mathrm{U}(A) \cap \mathcal{P}_1\}$ and let F' be the subgroup of $\mathrm{U}(q'Aq')$ generated by $\mathcal{V}'$. Let $\bar{F}'$ be the image of F' in $\mathrm{U}(q'Aq')/\mathrm{CU}(q'Aq')$. By 6.6(3) of [18], we have $\bar{F}' = (\bar{F}' \cap \mathrm{U}_0(q'Aq')/\mathrm{CU}(q'Aq')) \oplus \bar{F}_0' \oplus \bar{F}_1'$, where $\bar{F}_0'$ is torsion and $\bar{F}_1'$ is torsion free. Furthermore, we can assume $\mathcal{V}' = \mathcal{V}_0 \cup \mathcal{V}_1$ with $\bar{\mathcal{V}}_0$ generates $(\bar{F}' \cap \mathrm{U}_0(q'Aq')/\mathrm{CU}(q'Aq')) \oplus \bar{F}_0'$ and $\bar{\mathcal{V}}_1$ generates $\bar{F}_1'$.

Let $\mathcal{G}_2'$ be a finite subset of A which contains $\mathcal{S}_1, \mathcal{G}_{01}, L_1(\mathcal{G}_1'), H_1, \mathcal{V}', \{q', p_1', \cdots, p_n'\}$ and a finite generating set of S_2. By Theorem 3.24, a $\mathcal{G}_2' - \delta_1/4$ multiplicative completely positive linear contraction $\Phi_1' : A \to B$ such that

$$[\Phi_1]|_{\mathcal{P}_1 \cup \mathcal{P}_{01}} = \alpha|_{\mathcal{P}_1 \cup \mathcal{P}_{01}}$$

and

$$|\theta(\tau)(a) - \tau(L_1(a))| \leq \sigma/2 \text{ for all } a \in L_1H \cup H_1, \tau \in T(B).$$

We also assume $(\Phi_1')^\ddagger$ is well defined on $\bar{F}'$, $(\Phi_1' \circ L_1)^\ddagger$ is well defined on $\bar{F}$ and the image of Φ_1' is contained in B_n.

Let $B_n' = qB_nq$. Since B is simple, we may assume the rank of q is sufficiently large (> 6). By the construction , we have $[\Phi_1' \circ L_1](q)$ is equivalent to q. Therefore we may assume

$$||\Phi_1' \circ L_1(q) - q|| < \delta/4$$

by adjoining a unitary.

Write $B_n = \bigoplus_{j=1}^m B_n(j)$, where each $B_n(j)$ is a splitting interval algebra or the homogenous algebra with dimension less than 3. Therefore, we can write $q = q_1 \oplus q_2 \oplus \cdots \oplus q_l$ with $0 \leq l \leq m$ and $q_j \neq 0$. Choose an integer $N_1 > 0$ such that $N_1[q_j] \geq 3[1_{B_n(j)}$. Note that we assume q_j has rank at least 6. By applying an inner automorphism, we may assume that $\bigoplus_{j=1}^l B_n(j)$ is a hereditary C*-sub-algebra of $M_{N_1}(B_n')$. Since F_1 is finite generated, with sufficiently large n, we obtain a homomorphism $j : \bar{F}_1 \to \mathrm{U}(qB_n'q)/\mathrm{CU}(qB_n'q)$ such that $\phi_n^\ddagger \circ j = \mathrm{id}_{\bar{F}_1}$. Then

$$\kappa_1 \circ \phi_n^\ddagger \circ (\Phi_1' \circ L_1)^\ddagger|_{\bar{F}_1} = \kappa_1 \circ \phi_n^\ddagger \circ j = \kappa_1|_{\bar{F}_1},$$

where $\kappa_1 : \mathrm{U}(qBq)/\mathrm{CU}(qBq) \to \mathrm{K}_1(qBq)$ is the quotient map. Note that $\mathrm{K}_1(qBq) = \mathrm{K}_1(B)$. Let Δ_1 and δ be as in 5.13. We may assume that $\Delta_1 < \sigma_1/4$. To simplify notation, we assume that $\phi_n(q) = q$. By the assumption on B, we may write that $\phi_n|_{B_n'} = (\phi_n)_0 \oplus (\phi_n)_1$, where

(1) $\tau((\phi_n)_0(1_{B_n'}) < \delta_1/2(N_1+1)^2$ for all $\tau \in T(B)$ and

(2) $(\phi_n)_0$ is homotopically trivial (but non-zero).

It follows from Lemma 5.13 that there is a homomorphism $h : B'_n \to e_0Be_0$ such that

(1) $[h] = [(\phi_n)_0]$ in $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(B'_n), \underline{\mathrm{K}}(B))$ and
(2) $(\phi_n^\ddagger \circ j(\bar{(}w))^{-1}(h \oplus (\phi_n)_1)^\ddagger(\Lambda^\ddagger \bar{(}w)) = \bar{g}_w$, where $g_w \in \mathrm{U}_0(qBq)$ and $\mathrm{cel}(g_w) < \varepsilon/4$ (in $\mathrm{U}(qBq)$) for all $w \in \mathcal{U}_1$.

Define (we have assume that $B_n \subset M_{N_1}(B'_n)$)

$$h' = (h \oplus (\phi_n)_1 \otimes \mathrm{id}_{M_{N_1}})|_{\bigoplus_{j=1}^{l} B_n(j)},$$

and define $\Phi' = h' \oplus (\phi_n)|_{\oplus_{j=l+1}^{m} B_n(j)}$. Let $\Phi_1 = \Phi' \circ \Phi'_1$, we have

$$[\Phi_1]|_{\mathcal{P}_1 \cup \mathcal{P}_{01}} = [\Phi'_1]|_{\mathcal{P}_1 \cup \mathcal{P}_{01}} \text{ and } |\tau \circ \Phi_1(a) - \tau \circ \Phi'_1(a)| < \sigma_1/2$$

for all $a \in A_{s.a.}$ and $\tau \in \mathrm{T}(B)$. For all $w \in \mathcal{U}_1$, we have

$$\mathrm{cel}(w^*(\Phi_1 \circ L_1(w))) < 8\pi + \varepsilon/4 \text{ in } \mathrm{U}(qBq).$$

For any $w \in \mathcal{U}_0$, we also have

$$\mathrm{cel}(w^*(\Phi \circ L_1(w))) < 2\mathrm{cel}(w) + \pi/64 \text{ or } < 8\pi + 2\mathrm{cel}(w^k)/k + \pi/16$$

in $\mathrm{U}(qBq)$, depending in $[w] = 0$ or $[w]$ has order k in $\mathrm{K}_1(B)$. Therefore

$$\mathrm{cel}(\mathrm{id}_B(h_0(u))^{-1}(\Phi_1 \circ L_1(h_0(u))) < \mathbb{L}(u) \text{ in } \mathrm{U}(qBq)$$

for all $u \in \mathrm{U}(B) \cap \mathcal{P}_1$. Since we also have

$$[\mathrm{id}]|_{\mathcal{P} \cup \mathcal{P}_0} = [\Phi_1 \circ L_1]|_{\mathcal{P} \cup \mathcal{P}_0} \text{ and } \sup_{\tau \in T(B)} |\tau(a) - \tau(\Phi_1 \circ L_1(a))| < \delta$$

for all $a \in H$, by Theorem 4.6, there is a unitary $W \in \mathrm{U}(B)$ such that

$$||W \circ \Phi_1 \circ L_1(x)W^* - x|| < \varepsilon/2 \quad \text{for any } x \in \mathcal{F}.$$

Let $\mathcal{F}_2 \subset B$. We may assume $\mathcal{F}_2 \subset B_{m'_1}$ $(m'_1 > n)$. Let $\delta'_2 > 0$, the natural number n_2, finite subset $\mathcal{P}_2 \subset P(B)$, finite subsets $\mathcal{S}_2 \subset B$ as required in Theorem 4.6. Then there exist mutually orthogonal projections $q'', p''_1, \cdots, p''_n$ with $q'' \preceq p''_1$ and $p''_1, \cdots, p''_n$ mutually unitary equivalent, a C*-subalgebra S'_3 which is a splitting interval algebra with $1_{S'_3} = p''_1$ and unital $\mathcal{S} - \delta'_2/4$-multiplicative completely positive contractions $h''_0 : B \to q''Bq''$ with $h''_0(x) = q''xq''$, and $h''_1 : B \to S'_3$ such that

$$||x - h''_0(x) \oplus (\underbrace{h''_1(x) \oplus \cdots \oplus h''_1(x)}_{n \text{ copies}})|| \le \delta'_2/16$$

for all $x \in \mathcal{S}_2$. Put $S_2 = M_{n_2}(S'_3) \subset (1 - q'')B(1 - q'')$. Let $\mathcal{P}_{02}, \mathcal{G}_{02}, H_2, \delta_{02}$ and $\sigma_2 > 0$ be required by Theorem 4.6. Set $\delta_2 = \min\{\delta'_2, \delta_{02}\}$. We may assume that $\sigma_2 < \sigma_1/4$, $\delta_2 < \delta_1/4$, $[\Phi(\mathcal{P}_1 \cup \mathcal{P}_{01}) \subset [\mathcal{P}_2]$ and $\mathcal{P}_{02}$ contains the minimal projections of S_2 which present a generating set of the positive cone of $\mathrm{K}_0(S_2)$. Furthermore, we may assume that each $u \in \mathrm{U}(B) \cap \mathcal{P}_2$ has the form $q''uq'' \oplus (1 - q'')u(1 - q'')$, where $q''uq'' \in \mathrm{U}(q''Bq'')$ and $(1 - q'')u(1 - q'') \in \mathrm{U}(S_2)$. Put $\mathcal{W} = \{q''uq'' : u \in \mathrm{U}(B) \cap \mathcal{P}_2\}$. Let F'' be the subgroup generated by $\mathcal{W}$. Write $\bar{F}'' = (\bar{F}'' \cap \mathrm{U}_0(q''Bq'')/\mathrm{CU}(q''Bq'')) \oplus \bar{F}''_0 \oplus \bar{F}''_1$, where $\bar{F}''_0$ is torsion and $\bar{F}''_1$ is torsion free. We may also assume that $\Phi_1^\ddagger(\bar{F}') \subset \bar{F}''$. Furthermore, we also assume that $\mathcal{W} = \mathcal{W}_0 \cup \mathcal{W}_1$ where $\mathcal{W}_\prime$ generates $\bar{F}'' \cap \mathrm{U}_0(q''Bq'')/\mathrm{CU}(q''Bq'') \oplus \bar{F}_0''$ and $\mathcal{W}_1$ generates $\bar{F}_1''$.

Let $\mathcal{G}_3'$ be a finite subset which contains $\mathcal{S}_2, \mathcal{G}_{02}, q'', p_1'', \cdots, p_n'', H_2, \Phi_1(\mathcal{G}_2)$, a generating set of S_2 and $\mathcal{W}$. Without losing of generality, we can assume $\Phi_1(A) \subset B_m$ for some $m > m'$, and there is a completely positive linear map $J : B \to B_m$ such that

$$||J(a) - \mathrm{id}(a)|| < \delta_2/8.$$

Then we can find a projection $\tilde{p}' \in B_m$ such that

$$||\phi(q') - \tilde{q}'|| < \delta_2/2.$$

We may write $B_m = \oplus_{j=1}^s B_m(j)$. By choosing large m, we may also assume that $\tilde{q}'$ has at least rank 6. We write $\tilde{q}' = q_1' \oplus \cdots \oplus q_l'$ according to the direct sum decomposition ($q_i' \neq 0$ for each i). Let $N_2 > 0$ be an integer such that $N_2[q_j'] > 3[1_{m(j)}]$ for any j. Set $B_m' = \tilde{q}' B_m \tilde{q}'$. Note $\Phi_1^\ddagger$ is one-to-one on $\bar{F}_1'$. We may further assume that $\mathcal{G}_3'$ contains $q_1', \cdots, q_l'$ and a generating set of B_m' and B_m.

Now, let $L_2' : B \to A$ be a $\mathcal{G}_3' - \delta_2/16(N_2+1)^2$-multiplicative completely positive linear contraction such that

$$[L_2']|_{\mathcal{P}_2} = \alpha^{-1}|_{\mathcal{P} \cup \mathcal{P}_{02}} \quad \text{and}$$

$$\sup_{\tau \in TA} \{|\tau(L_2'(a)) - \theta^{-1}(\tau)(a)|\} < \delta_2/4 \quad \text{for any } a \in H_2 \cup \Phi_1(H_1).$$

We may assume $(L_2')^\ddagger$ is well defined on $\bar{F}''$. Let $e \in A$ is a projection such that

$$||L_2' \circ \Phi_1(q') - e|| < \delta_2/4.$$

Since $q' \in \mathcal{P}_1$, $[e] = [q']$ in $\mathrm{K}_0(A)$. Therefore, we can assume $e = q' = L_2' \circ \Phi_1(q')$ by adjoining some unitary in A. Note that $\bar{F}_1'$ is free and $(\phi_{m,M})_{*1}$ is one-to-one. We get that

$$\alpha^{-1} \circ (\phi_m)_{*1} \circ \kappa_1' \circ (\Phi_1)^\ddagger(g) = \kappa_1(g)$$

for all $g \in \bar{F}_1'$, where $\kappa_1' : \mathrm{U}(B_m')/\mathrm{CU}(B_m') \to \mathrm{K}_1(B_m')$ and $\kappa_1 : \mathrm{U}(\tilde{q}'B\tilde{q}')/\mathrm{CU}(\tilde{q}'B\tilde{q}') \to \mathrm{K}_1(\tilde{q}'B\tilde{q}')$ are the quotient maps. Note that we have $\mathrm{K}_1(\tilde{q}'B\tilde{q}') = \mathrm{K}_1(B)$. By Lemma 5.11, there exists a homomorphism $\beta : \mathrm{U}(B_m')/\mathrm{CU}(B_m') \to \mathrm{U}(q'Aq')/\mathrm{CU}(q'Aq')$ with $\beta(\mathrm{U}_0(B_m')/\mathrm{CU}(B_m')) \subset \mathrm{U}_0(q'Aq')/\mathrm{CU}(q'Aq')$ such that

$$\beta \circ (\Phi_1^\ddagger)(\bar{w}) = \bar{w}$$

for all $\bar{w} \in \bar{F}_1'$. Let $\delta_2' = \delta(\varepsilon/16)$. By the assumption on B, there is $M > m$ such that $\phi_{m,M} = \phi_{m,M}^{(0)} \oplus \phi_{m,M}^{(1)} : B_m \to B_M$ such that $\phi_{m,M}^{(0)}$ is homotopically trivial and $\tau(\phi_M \circ \phi_{m,M}^{(0)}(1_{B_m'})) < \Delta_2'/4(N_2+1)^2$ for all $\tau \in T(B)$. To simplify the notation, we assume that $e_0' = L_2' \circ \phi_M \circ \phi_{m,M}^{(0)}(1_{B_m'})$ and $e_1' = L_2' \circ \phi_M \circ \phi_{m,M}^{(1)}(1_{B_m'})$ are mutually orthogonal projections. It follows from Lemma 5.12 that there is a homomorphism $\Psi' : B_m' \to e_0' A e_0'$ such that

(1) Φ' is homotopically trivial, $\Phi_{*0}' = [L_2'] \circ (\phi_M \circ \phi_{m,M}^{(0)})_{*0}|_{\mathrm{K}_0(B_m')}$ and
(2) $(\beta(\Phi_1^\ddagger(\bar{w})^{-1})(\Phi' \oplus (L_2' \circ \phi_M \circ \phi_{m,M}^{(1)}))^\ddagger(\Phi_1^\ddagger(\bar{w})) = \bar{g}_w$ where $g_w \in \mathrm{U}_0(q'Aq')$ and $\mathrm{cel}(g_w) < \varepsilon/4$ (in $\mathrm{U}(q'Aq')$) for all $w \in \mathcal{V}_1$.

As in the construction of the map from A to B, we have a homomorphism $\tilde{\Phi}' : B_m \to q'Aq'$ such that $\tilde{\Phi}'$ is homotopically trivial, $\tilde{\Phi}'_{*0} = [L'_2] \circ (\phi_M \circ \phi_{m,M})_{*0}$ and $\tilde{\Phi}'|_{B'_m} = \Phi'$. Define $L_2 = (\tilde{\Phi}' \oplus L'_2 \circ \phi_M \circ \phi^{(1)}_{n,M}) \circ J$. One can verify that

$$[L_2]|_{\mathcal{P}_2 \cup \mathcal{P}_{02}} = [L'_2]|_{\mathcal{P}_2 \cup \mathcal{P}_{02}} = \alpha^{-1}|_{\mathcal{P}_2 \cup \mathcal{P}_{02}} \quad \text{and}$$

$$|\tau \circ L_2(a) - \tau \circ L'_2(a)| < \sigma_2/4$$

for all $a \in A_{s.a}$ with norm 1 and $\tau \in T(A)$. In particular

$$\sup_{\tau \in T(A)} \{|\tau \circ L_2 \circ \Phi_1(a) - \tau(a)|\} < \sigma_1/2$$

for all $a \in H_1$. Since $\beta \circ \Phi_1^{\ddagger}(\bar{w}) = \bar{w}$ for all $w \in \mathcal{V}_1$, we have

$$\mathrm{cel}(\mathrm{id}_A(h'_0(w^*))L_2(\Phi_1(h'_0(w)))) < 8\pi + \mathrm{cel}(g_w) + \varepsilon/4 < 8\pi + \varepsilon/2$$

in $\mathrm{U}(q'Aq')$ for all $w \in \mathcal{V}_1$. We also have

$$\mathrm{cel}(\mathrm{id}_A(h'_0(w^*))L_2(\Phi_1(h'_0(w))) < 2\mathrm{cel}((w) + \pi/16 \quad \text{or} \quad < 8\pi + 2\mathrm{cel}(w^k)/k + \pi/16$$

in $\mathrm{U}(q'Aq')$ for all $w \in \mathcal{V}_0$ (depends on $[w] = 0$ or has torsion k in $\mathrm{K}_1(A)$). Therefore, we have

$$\mathrm{cel}(\mathrm{id}(h'_0(u^*))L_2(\Phi_1(h'_0(u)))) < \mathbb{L}(u)$$

for all $u \in \mathrm{U}(A) \cap \mathcal{P}_2$ in $\mathrm{U}(q'Aq')$. By Theorem 4.6, we have a unitary $Z \in \mathrm{U}(A)$ such that

$$||Z(L_2 \circ \Phi(a))Z^* - a|| < \varepsilon/16 \quad \text{for all} \quad x \in \mathcal{F}_1.$$

Therefore, by replacing L_2 by $\mathrm{ad} \circ L_2$, we obtain the approximate intertwining diagram. By applying Elliott's intertwining argument, we see that A is isomorphic to B, and the isomorphism induces the given isomorphism between the invariants. □

6. A remark on the range of the invariant

Let $\{G, G^+, u\}$ be a scale-ordered group. Denote by $\mathrm{S}_u(G)$ be the convex of states on G, i.e.,

$$\mathrm{S}_u(G) = \{\rho : G \to \mathbb{R} : \ \rho(G^+) \subset \mathbb{R}^+, \rho(u) = 1\}.$$

The convex $\mathrm{S}_u(G)$ is compact with the pointwise convergence topology.

The range of the invariants of the class of TAS-algebras is strictly larger than the class of AH-algebras. For instance, it follows from [9] that there exists simple inductive limit A of splitting interval algebras such that $\mathrm{S}_u(\mathrm{K}_0(A))$ is a square, and hence $\mathrm{K}_0(A)$ is not a Riesz group. However, the class of TAS-algebra still does not exhaust all possible invariants.

Theorem 6.1. *Let G be a scale-order group. If $G \cong \varinjlim(G_i, \varphi_i)$, then $\mathrm{S}_u(G) \cong \varprojlim(\mathrm{S}_u(G_i), (\widehat{\varphi}_i))$.*

Lemma 6.2. *Let S be a splitting interval algebra. Denote by Δ the convex set of the states on the scaled-order $K_0(S)$. For any n vertices $\{e_1, ..., e_n\}$ of Δ, one has*

$$n - [\frac{n}{4}] - 1 \leq \dim(\mathrm{conv}\{e_1, ..., e_n\}).$$

Proof. Since S is a nuclear C*-algebra, any state on $\mathrm{K}_0(S)$ comes from a tracial state of S. Then, each extreme state corresponds to the Dirac measure concentrated on a splitting point. For the given extreme points $\{e_1, ..., e_n\}$, which corresponds to certain splitting points of S, the number of the relations them is at most $[\frac{n}{4}]$. Note that for an n-simplex, each such relation on the vertices reduces the dimension by 1. Thus, one has

$$\dim(\mathrm{conv}\{e, ..., e_n\}) \geq n - [\frac{n}{4}] - 1,$$

as desired. □

Corollary 6.3. *Let A be a simple TAS-algebra. Denote by Δ the convex $\mathrm{S}_u(K_0(A))$. Then, for any finite extreme points $\{e_1, ..., e_n\} \subseteq \Delta$, one has*

$$n - [\frac{n}{4}] - 1 \leq \dim(\mathrm{conv}\{e_1, ..., e_n\}).$$

In particular, if the number of the vertices of Δ is finite, denoted by m, one then has

$$m - [\frac{m}{4}] - 1 \leq \dim(\Delta).$$

Proof. By Theorem 6.1, one has the decomposition

$$\Delta \cong \varprojlim(\mathrm{S}_u(\mathrm{K}_0(S_i), \varphi_i)$$

for some splitting interval algebra S_i. For sufficiently large i, there is a map $\varphi_i : \Delta \to \mathrm{S}_u(\mathrm{K}_0(S_i))$ such that $\{\varphi_i(e_1), ..., \varphi_i(e_n)\}$ are distinct vertices of $\mathrm{S}_u(\mathrm{K}_0(S_i))$. By Lemma 6.2, one has

$$n - [\frac{n}{4}] - 1 \leq \dim(\varphi_i(\mathrm{conv}\{e_1, ..., e_n\})) \leq \dim(\mathrm{conv}\{e_1, ..., e_n\}),$$

as desired. □

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